BOARD OF STUDIES
NEW SOUTH WALES

MATHEMATICS
4 UNIT
Year 12

◆

Syllabus

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Mathematics K–12 Statement of Principles

Preface

This K–12 Statement of Principles describes the general principles that underlie all ideas in this syllabus.

The Nature of Mathematics

Mathematics is the study of number and space

More particularly it is:

- a search for patterns and relationships. This search, utilising acquired knowledge and skills, leads to the development of concepts and generalisations, which can be applied in finding solutions to problems, improving our understanding of the world around us and meeting the specific needs of people;
- a way of thinking, characterised by processes such as exploring, manipulating, discovering, ordering, classifying, generalising, abstracting, estimating, calculating, predicting, describing, deducing, drawing and measuring;
- a powerful, precise and concise means of communication, used to represent, to interpret, to explain and to predict;
- a creative activity. Accordingly, it involves invention, intuition and discovery.

Rationale for Mathematics in the Curriculum

Mathematics is useful

- Mathematics is essential for living. Some aspects of mathematics are required by individuals in order to function adequately as members of society. These aspects include strategies, skills and techniques involved in number facts, computation, mathematical problem-solving and reasoning.
- Mathematics is important and useful in many fields of endeavour. These fields include the sciences, medicine, economics, commerce, industry, engineering, business and the arts.
- Mathematics provides a means of oral and written communication. Mathematics can be used to present and convey information in many ways. Some of these include explanations, figures, letters, tables, charts, diagrams, graphs and drawings.
- Mathematics provides opportunities for development of reasoning abilities.

Mathematics is part of our culture

Mathematics has been part of human activity since the earliest times. It has made, and continues to make, a significant contribution to human culture. Mathematics allows students to appreciate their cultural heritage more fully by providing insights into many of the creative achievements of the human race.

Mathematics can be part of our leisure

Mathematics is a source of interesting and appealing puzzles and problems. When mathematics is enjoyable, it encourages curiosity, exploration, discovery and invention.

Aims of Mathematics Education

The aims of mathematics education will be achieved in different levels in the K–12 range according to the stages of development of the students at these levels. These aims are to develop in students confidence and enjoyment in doing mathematical activities, knowledge, skills and understandings in certain specified areas, and awareness of the place of mathematics in solving problems of everyday life and in contributing to the development of our society.

Confidence and enjoyment in doing mathematical activities

- Confidence in their ability to do mathematics.
- A positive attitude to mathematics as an interesting, enjoyable and challenging subject.
- An appreciation of mathematics as a creative activity with aesthetic appeal.

Knowledge, skills and understanding in certain specified areas

- Thinking that is logical, flexible, fluent and original.
- Skills in computation and problem solving in all areas of mathematics.
- Appropriate language for the effective communication of mathematical ideas and experiences.
- An ability to recognise mathematical patterns and relationships.
- A variety of methods for calculations and problem-solving.
- An awareness of the basic structure of mathematics by an appreciation of the nature and interrelationship of the various strands of mathematics.

Awareness of the place of mathematics in solving problems of everyday life and in contributing to the development of our society

- An ability to apply mathematical ideas, rules and procedures to particular situations and problems.

- An awareness that the learning of mathematics includes the processes of inquiry, discovery and verification.
- An awareness of the uses of mathematics both in and beyond the classroom.
- An appreciation of mathematics as a relevant and useful activity.
- An appreciation of appropriate uses of technology, including calculators and computers.

The Nature of Mathematics Learning

Students learn best when motivated

Mathematics learning is more effective when it is interesting, enjoyable and challenging.

Implications

- Learning activities should provoke curiosity, should be appropriate to students' stages of development, and should be related to everyday life experiences.
- The actual experiences of students should be used as the source of many learning activities.
- Students should be encouraged to become aware of the relevance of mathematics to their lives.
- Students should often experience success in mathematical activities. A positive attitude towards mathematics and towards oneself will be promoted by emphasising the students' achievements.

Students learn mathematics through interacting

Mathematics learning should involve interaction with the physical and social environment, leading to the abstraction of particular mathematical ideas encountered.

Implications

- The understanding of mathematical ideas is promoted by interaction with people and manipulation of materials in a wide variety of learning situations.
- Cooperative learning in small groups provides excellent opportunities for interaction.

Mathematics learning is promoted by the appropriate use of a variety of materials, equipment and personnel.

Implications

- Materials and equipment should be used in imaginative ways to explore, discover and develop mathematical ideas.

- The availability of technological equipment, such as calculators and computers, does not reduce the need for mathematical understanding or the need for competence.

- Some concepts and skills will need to receive greater emphasis with the introduction of calculators and computers, eg place value and decimal concepts; skills of approximation and estimation.

Students learn mathematics through investigating

Mathematics learning should involve the investigation of mathematical patterns, relationships, processes and problems.

Implications

- Students should be given opportunities to discover and create patterns, and to describe and record relationships contained in those patterns.

- Opportunities to use mathematical processes and to compose and solve problems should be provided in all strands of mathematics.

Students learn mathematics through language

Mathematics learning is promoted by the appropriate use of language. Language, including symbols and diagrams, plays an important part in the formulation and expression of mathematical ideas and serves as a bridge between concrete and abstract representation.

Implications

- Mathematical activities should be regarded as opportunities for teachers and students to use and develop appropriate language.

- It is important that teachers be familiar with the language patterns appropriate to the different mathematical processes.

- Students should be encouraged to use oral and written language appropriate to their particular stage of development to gain meaning from their mathematical learning experiences.

- When developing teaching strategies and learning activities in mathematics, teachers should give consideration to the diverse cultural and linguistic backgrounds of students.

Students learn mathematics as individuals but in the context of intellectual, physical and social growth

Mathematics learning is promoted when individual differences of students are taken into account.

Implications

- Students vary in the way and the rate in which they learn mathematics.
- Learning experiences should be appropriate to the student's stage of development.
- Teachers should take into account the student's knowledge gained formally and informally outside the school, including the home.
- Recognition should be given that the whole of society has mathematical ability. Maximum participation and extension of all students, regardless of sex, is appropriate.

Mathematics learning should be appropriate to each student's current stage of development and should build upon previous experiences and achievements.

Implications

- To cater for the variety of developmental levels that may exist among a group of students, teachers should provide a flexible learning situation where there is a variety of opportunities for involvement.
- Whilst the student's readiness to proceed to new work will depend on previous knowledge and understanding, this does not mean that there is an absolute order in which mathematics learning should proceed for all students. There are many paths to understanding.
- Teachers should respond to emergent opportunities to capitalise on the student's interests and needs and vary the intended sequence of mathematical experience.
- As each new mathematical concept is encountered, learning should proceed, where possible, from the concrete to the abstract. Concepts should be continually developed and consolidated through a wide variety of learning experiences.
- The development of understanding should, as a general principle, precede a requirement for both automatic recall of factual information and speed and accuracy in performing mathematical computations. Skills should be maintained through meaningful practice and enjoyable drill.

Assessment and Evaluation

In implementing any syllabus based on this Statement of Principles, schools must be careful to evaluate the program offered by the school and to assess the progress of individuals within that program.

The Mathematics 4 Unit Syllabus

Introduction

The Mathematics 4 Unit course is defined in the same terms as the 3 Unit Course in other subjects. Thus it offers a suitable preparation for study of the subject at tertiary level, as well as a deeper and more extensive treatment of certain topics than is offered in other Mathematics courses.

This syllabus is designed for students with a special interest in mathematics who have shown that they possess special aptitude for the subject. It represents a distinctly high level in school mathematics involving the development of considerable manipulative skill and a high degree of understanding of the fundamental ideas of algebra and calculus. These topics are treated in some depth. Thus the course provides a sufficient basis for a wide range of useful applications of mathematics as well as an adequate foundation for the further study of the subject.

Aims and Objectives

The Board of Secondary Education recognises that the aims and objectives of the syllabus may be achieved in a variety of ways and by the application of many different techniques. Success in the achievement of these aims and objectives is the concern of the Board which does not, however, either stipulate or evaluate specific teaching methods.

The general aim is to present mathematics as a living art which is intellectually exciting, aesthetically satisfying, and relevant to a great variety of practical situations.

Specific aims of the course are:

- To offer a program that will be of interest and value to students with the highest levels of mathematical ability at the stage of the Higher School Certificate and which will present some challenge to such students.
- To study useful and important mathematical ideas and techniques appropriate to these levels of ability.
- To develop both an understanding of these ideas and techniques and an ability to apply them to the study and solution of a wide variety of problems.
- To provide the mathematical background necessary for further studies in mathematics, and useful for concurrent study of subjects such as science, economics and industrial arts.

Syllabus Structure

The objectives of this syllabus are addressed through eight topics:

- Graphs
- Complex Numbers
- Conics
- Integration
- Volumes
- Mechanics
- Polynomials
- Harder 3 Unit topics.

It is not intended that equal amounts of school time be allocated to the topics and there is no mandatory sequence. However, approximately 30% of time should be devoted to harder 3 Unit topics.

While the Board is unable to make implementation strategies compulsory, it is expected that, for the sake of relevance and student interest, this syllabus will involve

- applications
- proofs
- problem solving
- use of computers and calculators
- practical experiments.

Assessment

This section provides some general information on activities suitable for both formative and summative assessment. The Higher School Certificate assessment requirements are set out in the HSC Assessment Guide.

Assessment forms an integral and continuous part of any teaching program. The purposes of assessment include:

- the identification of students' needs
- the measurement of students' achievements
- the measurement of the results of a course of action.

Both student and teacher should be aware of the purpose of an assessment task. Ongoing (formative) assessment, end-of-unit (summative) assessment and ranking require different assessment strategies. The strategies suitable for the purpose of formative assessment can include observation of student work and informal discussion as well as tests of varying length. Written tests would be major contributions to summative assessment in this course.

Assessment of student achievement in relation to the objectives of the Syllabus should incorporate measures of:

- knowledge
- operational facility
- comprehension
- communication
- applications in realistic situations
- analysis of situations leading to solution of problems
- mathematical proofs.

Evaluation

The evaluation of programs of learning based on this Syllabus should involve students as well as teachers. The evaluation should encompass:

- student attitudes, assessed informally and/or through a questionnaire;
- teaching strategies, evaluated in terms of student morale and coverage of course objectives;
- resources, evaluated in terms of availability and effectiveness;
- parent and community reaction, assessed through informal contacts and/or by whatever available structures exist, eg meeting of parent organisations;
- relevance, as determined by students both past and present.

This will involve qualitative as well as quantitative measures.

Topic 1

Graphs

Topic 1: Graphs

Contents and Skills Objectives

1.1 Basic Curves

The student is able to:

- graph a linear function ($ax + by + c = 0$, $y = mx + b$)
- graph a quadratic function ($y = ax^2 + bx + c$)
- graph a cubic function ($y = ax^3 + bx^2 + cx + d$)
- graph a quartic function ($y = ax^4 + bx^3 + cx^2 + dx + e$)
- graph a rectangular hyperbola ($xy = k$)
- graph a circle ($x^2 + y^2 + 2gx + 2fy + c = 0$)
- graph an exponential function ($y = a^x$ for both cases $a > 1$ and $0 < a < 1$)
- graph a logarithmic function ($y = \log_a x$)
- graph trigonometric functions (eg $y = k + a\sin(bx + c)$)
- graph inverse trigonometric functions (eg $y = a\sin^{-1}bx$)
- graph the functions $y = x^{1/2}$ and $y = x^{1/3}$.

1.2 Drawing graphs by addition and subtraction of ordinates

The student is able to:

- graph a function $y = f(x) \pm c$ by initially graphing $y = f(x)$
- graph a function $y = f(x) \pm g(x)$ by initially graphing $y = f(x)$ and $y = g(x)$.

1.3 Drawing graphs by reflecting functions in coordinate axes

The student is able to:

- graph $y = -f(x)$ by initially graphing $y = f(x)$
- graph $y = |f(x)|$ from the graph of $y = f(x)$
- graph $y = f(-x)$ by initially graphing y = $f(x)$.

Topic 1: Graphs

Applications, Implications and Considerations

- The initial emphasis in this topic is operating on graphs of these basic functions in order to produce a graph of a more complex function (eg the graph of $y = x \log_e x$ will be developed by considering properties of the graphs of $y = x$ and $y = \log_e x$).

- The notations $\log_e x$ and $\ln x$ are used to denote the natural logarithm of x and students should be familiar with both notations.

- Students will need to be able to produce quickly a neat sketch of these basic functions in order to use them in the sketching of further functions.

- Students need to examine the behaviour of the derivatives of $y = x^{1/2}$ and $y = x^{1/3}$ near $x = 0$ and investigate the behaviour of these functions at $x = 0$. They must be familiar with the term 'critical point' and with the possibility of curves having vertical tangent lines at points on them.

- Typical functions involving addition of ordinates could include $y = 1 + 3 \sin 2x$ for $-2\pi \le x \le 2\pi$ and $y = \cos^{-1} x - \pi$. Students should realise that the graph of $3 \sin 2x$ can be transformed to the graph of $1 + 3 \sin 2x$ by either translating the graph one unit upwards or translating the x-axis one unit in the opposite direction.

- Other types could include graphing functions such as $f(x) = 3 \sin x + x$ for $0 < x < 4$. This may be developed from the graphs of $y = x$ and $y = 3 \sin x$. The points where $y = 3 \sin x$ cuts the x-axis correspond to the points where $y = 3 \sin x + x$ cuts $y = x$. Once the shape of the curve has been roughed out using addition of ordinates the position of stationary points and points of inflexion may be obtained when appropriate.

- A function such as $y = -\log_e x$ may be graphed by reflecting the graph of $y = \log_e x$ in the x-axis. The graph of $y = 2 - \log_e x$ may then be obtained by a suitable translation.

- The relationship between the graphs of $y = f(x)$ and of $y = f(x - a)$ should be discussed and used also in examples involving the reflection properties, such as, for example, the graph of $|1 - \sin(x - 2)|$.

Topic 1: Graphs

Contents and Skills Objectives

1.4 Sketching functions by multiplication of ordinates

The student is able to:

- graph a function $y = cf(x)$ by initially graphing $y = f(x)$
- graph a function $y = f(x) \cdot g(x)$ by initially graphing $y = f(x)$ and $y = g(x)$.

1.5 Sketching functions by division of ordinates

The student is able to:

- graph a function $y = 1/f(x)$ by initially graphing $y = f(x)$
- graph a function $y = f(x)/g(x)$ by initially graphing $y = f(x)$ and $y = g(x)$.

Topic 1: Graphs

Applications, Implications and Considerations

- A good initial idea of the behaviour of functions of the form $f{\cdot}g$ may be obtained by examining the graphs of f and g independently.

  To graph $y = xe^{-x}$, the functions $y = x$ and $y = e^{-x}$ may be graphed on the same set of axes.

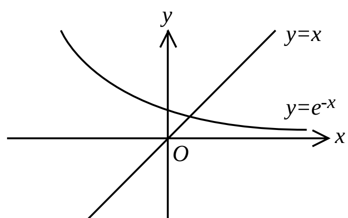


Fig 1

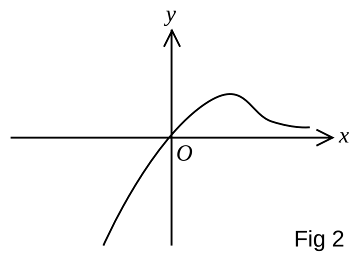


Fig 2

  From Figure 1, important features of the graph of $y = xe^{-x}$ can be obtained. These include properties that

  - for $x < 0$, $xe^{-x} < 0$; for $x = 0$, $xe^{-x} = 0$; for $x > 0$, $xe^{-x} > 0$
  - As $x \to -\infty$, $xe^{-x} \to -\infty$; as $x \to \infty$, $xe^{-x} \to 0$.

  This enables a rough shape to be quickly sketched (Figure 2). The exact positions of the stationary point and point of inflexion may be determined by calculus.

- $y = 1/f(x)$ may be sketched by first sketching $y = f(x)$. Where $f(x) = 0$, $1/f(x)$ is undefined; where $f(x) > 0$, $1/f(x) > 0$; where $f(x) < 0$, $1/f(x) < 0$. As well, when $f(x)$ is increasing then $1/f(x)$ is decreasing and vice versa.

- $y = f(x)/g(x)$ may be sketched by initially sketching $f(x)$ and $g(x)$. Where $f(x) = 0$, $f(x)/g(x) = 0$ and where $g(x) = 0$, $f(x)/g(x)$ is undefined and a discontinuity exists. Examination of the signs of $f(x)$ and $g(x)$ will lead to where $f(x)/g(x)$ is positive and where it is negative. These results can be used together with the approach to graphs which are products of functions to obtain an idea of the shape of a function. To graph $y = \frac{x(x+1)}{(x-2)}$, the graphs of $y = x$, $y = x + 1$ and $y = x - 2$ can be first sketched. The regions in the number plane, in which the graph exists, can be then shaded, discontinuities determined, points of intersection with coordinate axes marked on and the behaviour of the function for $x \to \pm\infty$ investigated. Exact positions of stationary points and points of inflexion could lastly be determined if required.

Topic 1: Graphs

Contents and Skills Objectives

1.6 Drawing graphs of the form $[f(x)]^n$

The student is able to:

- graph a function $y = [f(x)]^n$ by first graphing $y = f(x)$.

1.7 Drawing graphs of the form $\sqrt{f(x)}$

The student is able to:

- graph a function $y = \sqrt{f(x)}$ by first graphing $y = f(x)$.

1.8 General approach to curve sketching

The student is able to:

- use implicit differentiation to compute $\frac{dy}{dx}$ for curves given in implicit form
- use the most appropriate method to graph a given function or curve.

Topic 1: Graphs

Applications, Implications and Considerations

- The graph of $[f(x)]^n$ ($n > 1$, n an integer) may be drawn by sketching $y = f(x)$ and realising that, since its derivative is $n[f(x)]^{n-1} f'(x)$, then all stationary points and intercepts on the x-axis of $y = f(x)$ are stationary points of $[f(x)]^n$. Other features worth examining include the properties that if $|f(x)| > 1$, then $|[f(x)]^n| > |f(x)|$ and that if $0 < |f(x)| < 1$ then $0 < |[f(x)]^n| < |f(x)|$. Also if n is even, $[f(x)]^n \geq 0$ for all x but if n is odd, $[f(x)]^n > 0$ for $f(x) > 0$ and $[f(x)]^n < 0$ for $f(x) < 0$.

- The graph of $\sqrt{f(x)}$ may be developed from the graph of $f(x)$ by noting that
  - $\sqrt{f(x)}$ is defined only if $f(x) \geq 0$
  - $\sqrt{f(x)} \geq 0$ for all x in the natural domain
  - $\sqrt{f(x)} < f(x)$ if $f(x) > 1$, $\sqrt{f(x)} = f(x)$ if $f(x) = 1$, $\sqrt{f(x)} > f(x)$ if $0 < f(x) < 1$.
  - If $y = \sqrt{f(x)}$ then $y' = \dfrac{f'(x)}{2\sqrt{f(x)}}$. This leads to the position of stationary points and the existence of critical points (where y' is undefined).

- To sketch $y = \sqrt{\dfrac{x(x-1)}{x-2}}$, a rough sketch of $y = \dfrac{x(x-1)}{x-2}$ can first be drawn and then the above ideas utilised to sketch the function.

- The graph of a function could be obtained by using techniques which utilise basic functions together with consideration of features such as discontinuities, finding the domain of a function, investigating the behaviour of the function in the neighbourhood of $x = 0$, considering the behaviour of the function for x large, testing whether a function is odd or even, plotting points and deciding on positions of stationary points, critical points and points of inflexion.

- Curves graphed could include $y = \dfrac{x^4}{x^2 - 1}$, $y = x^2 e^{-x}$,

  $y = x \ln(x^2 - 1)$, $y^2 = x^2 - 9x$, $x^2 + 2y^2 = 4$, $y = \dfrac{\sin x}{x}$

  and $y = x \cos x$.

Topic 1: Graphs

Contents and Skills Objectives

1.9 Using graphs

The student is able to:

- solve an inequality by sketching an appropriate graph
- find the number of solutions of an equation by graphical considerations
- solve problems using graphs.

Topic 1: Graphs

Applications, Implications and Considerations

- Typical inequalities could include absolute values

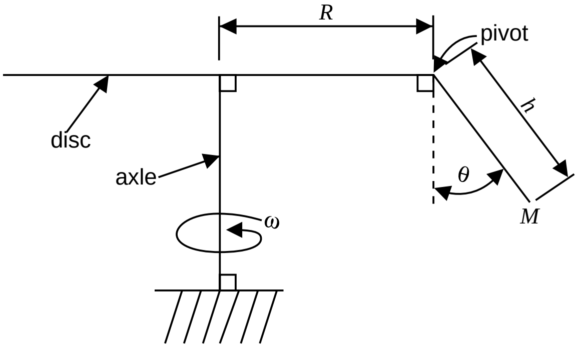
  ($|x + 2| + |x - 1| > 4$), rational functions $\left[\frac{x^2 - 1}{x^2 - 4} > 1\right]$ and trigonometric functions ($\sin 2x \geq \frac{1}{2}$).

  - The number of roots of an equation can be investigated graphically (eg find the number of solutions of $x^3 - kx^2 + k = 0$ for varying values of k).

  - Graphs often give elegant solutions to physical problems which would be difficult to solve by other means. (eg Question 6, 4 Unit HSC Mathematics paper, 1982.)

Question 6 *(1982)*

A 'chair-o-plane' at a fairground consists of seats hung from pivots attached to the rim of a horizontal circular disc, which is rotated by a motor driving its vertical axle. As the speed of rotation increases, the seats swing out. Represent a seat by a point mass M kg suspended by a weightless rod h metres below a pivot placed R metres from the axis of rotation. Assume that when the disc rotates about its axle with constant angular velocity ω radians per second, there is an equilibrium position in which the rod makes an angle θ with the vertical, as shown in the figure.



(a) Show that ω, θ satisfy the relation
$(R + h \sin \theta)\omega^2 = g \tan \theta$.

(b) Use a graphical means to show that, for given ω there is just one value of θ in the range $0 \leq \theta \leq \frac{\pi}{2}$ which satisfies this relation.

(c) Given $R = 6$, $h = 2$, $\theta = 60°$, and assuming that $g = 10$ ms^{-2}, find the speed of M relative to the ground.

Topic 2

Complex Numbers

Topic 2: Complex Numbers

Contents and Skills Objectives

2.1 Arithmetic of complex numbers and solving quadratic equations

The student is able to:

- appreciate the necessity of introducing the symbol i, where $i^2 = -1$, in order to solve quadratic equations
- write down the real part Re(z) and the imaginary part Im(z) of a complex number $z = x + iy$.
- add, subtract and multiply complex numbers written in the form $x + iy$
- find the complex conjugate $\bar{z}$ of the number $z = x + iy$
- divide a complex number $a + ib$ by a complex number $c + id$
- write down the condition for $a + ib$ and $c + id$ to be equal
- prove that there are always two square roots of a non-zero complex number
- find the square roots of a complex number $a + ib$
- solve quadratic equations of the form $ax^2 + bx + c = 0$, where a, b, c are complex.

Topic 2: Complex Numbers

Applications, Implications and Considerations

- Students could be introduced to i as a device by which quadratic equations with real coefficients could be always solvable. The arithmetic and algebra of complex numbers would then be developed and eventually it could be shown that there exist 2 complex roots for a complex number. This then leads to the discovery that a quadratic equation with complex coefficients will have 2 complex roots.

- In finding the square roots of $a + ib$, the statement $\sqrt{a+ib} = x + iy$, where a, b, x, y are real, leads to the need to solve the equations $x^2 - y^2 = a$ and $2xy = b$. Examining graphs of these curves for various values of a and b will lead to the conclusion that two roots will always exist for a complex number.

Topic 2: Complex Numbers

Contents and Skills Objectives

2.2 Geometric representation of a complex number as a point

The student is able to:

- appreciate that there exists a one to one correspondence between the complex number $a + ib$ and the ordered pair (a,b)
- plot the point corresponding to $a + ib$ on an Argand diagram
- define the modulus ($|z|$) and argument ($\arg z$) of a complex number z
- find the modulus and argument of a complex number
- write $a + ib$ in modulus-argument form

- prove basic relations involving modulus and argument
- use modulus-argument relations to do calculations involving complex numbers.

- recognise the geometrical relationships between the point representing z and points representing $\bar{z}$, cz (c real) and iz.

Topic 2: Complex Numbers

Applications, Implications and Considerations

- The geometrical meaning of modulus and argument for $z = x + iy$ should be given and the following definitions used:

  $$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2},$$

  arg z is any value of θ for which $x = |z| \cos\theta$ and $y = |z| \sin\theta$. While arg z is commonly assigned a value between $-\pi$ and π, the general definition is needed in order that the relations involving arg z given below are valid.

- Students should at least be able to prove the following relations:

  $|z_1 z_1| = |z_1| \cdot |z_2|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

  $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $\arg\left[\frac{z_1}{z_2}\right] = \arg z_1 - \arg z_2$

  $|z^n| = |z|^n$, $\arg(z^n) = n \arg z$

  $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$, $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$.

- The fact that multiplication by i corresponds to an anticlockwise rotation through $\pi/2$ about O, that $\bar{z}$ is the reflection of z in the real axis and that multiplication by c corresponds to an enlargement about O by a factor c, should be used on simple geometrical exercises. (See Question 4 (iii), 4 Unit HSC Mathematics paper, 1987).

Question 4. (iii) *(1987)*

(a) Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively. Find the complex number represented by B.

(b) The square is now rotated about O through 45° in an anticlockwise direction to $OA'B'C'$. Find the complex numbers represented by the points A', B' and C'.

Topic 2: Complex Numbers

Contents and Skills Objectives

2.3 Geometrical representations of a complex number as a vector

The student is able to:

- appreciate that a complex number z can be represented as a vector on an Argand diagram
- appreciate the geometrical significance of the addition of two complex numbers
- given the points representing z_1 and z_2, find the position of the point representing z, where $z = z_1 + z_2$
- appreciate that the vector representing $z = z_1 + z_2$ corresponds to the diagonal of a parallelogram with vectors representing z_1 and z_2 as adjacent sides
- given vectors z_1 and z_2, construct vectors $z_1 - z_2$ and $z_2 - z_1$
- given z_1 and z_2, construct the vector $z_1 z_2$
- prove geometrically that $|z_1 + z_2| \leq |z_1| + |z_2|$.

2.4 Powers and roots of complex numbers

The student is able to:

- prove, by induction, that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for positive integers n
- prove that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for negative integers n
- find any integer power of a given complex number
- find the complex n^{th} roots of ± 1 in modulus-argument form
- sketch the n^{th} roots of ± 1 on an Argand diagram
- illustrate the geometrical relationship connecting the n^{th} roots of ± 1.

Topic 2: Complex Numbers

Applications, Implications and Considerations

- Familiarity with the vector representation of a complex number is extremely useful when work on curves and loci is encountered.
- Students need to be able to interpret the expression $|z - (a + ib)|$ as the magnitude of a vector joining (a,b) to the point representing z.
- Students need to recognise that the expression $\arg(z - z_1)$ refers to the angle, which a vector joining the point representing z_1 to the point representing z, makes with the positive direction of the real axis.

- $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for negative integers n follows algebraically from the previous result.

- Students should realise that points corresponding to the n^{th} roots of ± 1 are equally spaced around the unit circle with centre O and so form the vertices of a regular n-sided polygon.

Topic 2: Complex Numbers

Contents and Skills Objectives

2.5 Curves and Regions

The student is able to:

- given equations $\text{Re}(z) = c$, $\text{Im}(z) = k$ (c, k real), sketch lines parallel to the appropriate axis
- given an equation $|z - z_1| = |z - z_2|$, sketch the corresponding line
- given equations $|z| = R$, $|z - z_1| = R$, sketch the corresponding circles
- given equations $\arg z = \theta$, $\arg(z - z_1) = \theta$, sketch the corresponding rays
- sketch regions associated with any of the above curves (eg the region corresponding to those z satisfying the inequality ($|z - z_1| \leq R$)
- give a geometrical description of any such curves or regions

- sketch and describe geometrically the intersection and/or union of such regions

- sketch and give a geometrical description of other simple curves and regions.

Topic 2: Complex Numbers

Applications, Implications and Considerations

- Typical curves and regions are those defined by simple equations or inequalities, such as

  $\mathrm{Im}(z) = 4$, $|z - 2 - 3i| = |z - i|$, $|z - 3 + 4i| = 5$, $\mathrm{Re}(z) > 2$,
  $0 < \arg z < \pi/2$, $0 \le \mathrm{Im}(z) \le 4$.

- Simple intersections, such as the region common to $|z| = 1$ and $0 \le \arg z \le \pi/4$, and corresponding unions, need be done.

- Examples need only involve replacing z by $z = x + iy$ in relations such as $2|z| = z + \bar{z} + 4$, $z + \bar{z} > 0$, $|z^2 - (\bar{z})^2| < 4$. They need not include discussion of curves such as $w = \frac{z - i}{z + i}$, where z lies on a unit circle.

Topic 3

Conics

Topic 3: Conics

Contents and Skills Objectives

3.1 The Ellipse

The student is able to:

- write down the defining equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of an ellipse with centre the origin
- sketch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, showing points of intersection with axes of symmetry
- find the lengths of the major and minor axes and semi-major and semi-minor axes of an ellipse
- write down the parametric coordinates $(a \cos\theta, b \sin\theta)$ of a point on an ellipse
- sketch an ellipse using its auxiliary circle
- find the equation of an ellipse from its focus-directrix definition
- find the eccentricity from the defining equation of an ellipse
- given the equation of an ellipse, find the coordinates of the foci and equations of the directrices
- sketch an ellipse, marking on the sketch the positions of its foci and directrices
- use implicit differentiation to find the equations of the tangent and the normal at $P(x_1, y_1)$ on an ellipse
- find the equations of the tangent and the normal at $P(a \cos\theta, b \sin\theta)$ on an ellipse
- find the equation of a chord of an ellipse
- find the equation of a chord of contact
- prove that the sum of the focal lengths is constant
- prove the reflection property, namely that the tangent to an ellipse at a point P on it is equally inclined to the focal chords through P
- prove that the chord of contact from a point on a directrix is a focal chord.

Topic 3: Conics

Applications, Implications and Considerations

- The cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is taken as the defining equation.
- The major properties of an ellipse are to be proven for both a general ellipse with centre O and for ellipses with given values of a and b.
- The parametric representation $x = a \cos\theta$, $y = b \sin\theta$ is useful in graphing the ellipse from an auxiliary circle. The shape of an ellipse should be examined as the ratio $\frac{b}{a}$ varies.
- The focus-directrix definition should be used whenever a focal distance is to be calculated.
- The focus-directrix definition leads to a simple proof that the sum of the focal lengths is constant.
  $(SP + S'P = e(PM + PM') = e \cdot \frac{2a}{e} = 2a.)$
- The chord of contact is useful as a tool in the proof of a number of properties of an ellipse.
- The reflection property of the ellipse may be approached by using the result that the bisector of an angle of a triangle divides the opposite side into two intervals, whose lengths are in the same ratio as the lengths of the other two sides.

Topic 3: Conics

Contents and Skills Objectives

3.1 *(continued)*

- prove that the part of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus
- prove simple properties for both the general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and for ellipses with given values of a and b.

Topic 3: Conics

Applications, Implications and Considerations

- Students are not expected to do proofs, under examination conditions, which are more difficult than those involved in the Contents and Skills objectives.
- Locus problems on the ellipse are *not* included.

Topic 3: Conics

Contents and Skills Objectives

3.2 The Hyperbola

The student is able to:

- write down the defining equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of a hyperbola with centre the origin
- sketch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, showing points of intersection with axes of symmetry and positions of asymptotes
- find the length of major and minor axes and semi-major and semi-minor axes of a hyperbola
- write down the parametric coordinates $(a \sec\theta, b \tan\theta)$ of a point on the hyperbola
- find the equation of a hyperbola from its focus-directrix definition
- find the eccentricity from the defining equation of a hyperbola
- given the equation of the hyperbola, find the coordinates of its foci and equations of its directrices
- sketch a hyperbola, marking on the positions of its foci and directrices
- use implicit differentiation to find the equations of the tangent and the normal at $P(x_1, y_1)$ on a hyperbola
- find the equations of the tangent and the normal at $P(a \sec\theta, b \tan\theta)$ on the hyperbola
- find the equation of a chord of a hyperbola
- find the equation of a chord of contact
- prove that the difference of the focal lengths is constant
- prove the reflection property for a hyperbola
- prove that the chord of contact from a point on the directrix is a focal chord
- prove simple properties for both the general hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and for hyperbolae with given values of a and b.

Topic 3: Conics

Applications, Implications and Considerations

- The major properties of the hyperbola are to be proven for both the general hyperbola with centre O and for hyperbolae with given values of a and b.

- The shape of the hyperbola should be examined as $\frac{b}{a}$ varies.

- The focus-directrix definition should be used whenever a focal distance is to be calculated.

- The chord of contact is useful in proving some properties.

- The same geometry theorem, as used in the case of the ellipse, is useful in proving the reflection property of the hyperbola.

- Students are not expected to do proofs, under examination conditions, which are more difficult than those involved in the skills objectives.
- Locus problems, on a hyperbola with equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (where $a \neq b$), are not in the course.

Topic 3: Conics

Contents and Skills Objectives

3.3 The Rectangular Hyperbola

The student is able to:

- prove that the hyperbola with equation $xy = \frac{1}{2}a^2$ is the hyperbola $x^2 - y^2 = a^2$ referred to different axes
- write down the eccentricity, coordinates of foci and vertices, equations of directrices and equations of asymptotes of $xy = \frac{1}{2}a^2$
- sketch the hyperbola $xy = \frac{1}{2}a^2$, for varying values of a, marking on vertices, foci, directrices and asymptotes
- write down the parametric coordinates $(ct, \frac{c}{t})$ for the rectangular hyperbola $xy = c^2$, for varying values of c
- find the equation of the chord joining $P(cp, \frac{c}{p})$ to $Q(cq, \frac{c}{q})$
- find the equation of the tangent at $P(cp, \frac{c}{p})$
- find the equation of the normal at $P(cp, \frac{c}{p})$
- find the equation of the chord joining $P(x_1, y_1)$ to $Q(x_2, y_2)$
- find the equation of the chord of contact from $T(x_0, y_0)$
- find the point of intersection of tangents and of normals
- prove simple geometrical properties of the rectangular hyperbola including:
  - the area of the triangle bounded by a tangent and the asymptotes is a constant
  - the length of the intercept, cut off a tangent by the asymptotes, equals twice the distance of the point of contact from the intersection of the asymptotes
- find loci of points including:
  - loci determined by intersection points of tangents
  - loci determined by intersection points of normals
  - loci determined by midpoints of intervals.

Topic 3: Conics

Applications, Implications and Considerations

- A definition needs to be given for a rectangular hyperbola. It quickly follows, from seeing the connection between $x^2 - y^2 = a^2$ and $xy = \frac{1}{2}a^2$, that the eccentricity is $\sqrt{2}$.

- Geometrical properties of the rectangular hyperbola should be proven for the rectangular hyperbola $xy = \frac{1}{2}a^2$ for varying values of a.

- It is not intended that locus problems should include sophisticated techniques for elimination of parameters. Students are expected to be able to proceed from a pair of parametric equations to obtain a locus expressible by a linear equation (perhaps with constraints on x or y). In cases where the resulting locus is not expressible in terms of a linear equation, it will be given in algebraic or geometric form and students will verify that this form is satisfied (perhaps with additional constraints).

  A good example of a harder locus problem is given by the last part of Question 5, 4 Unit HSC Mathematics paper, 1988 (see over).

Topic 3: Conics

Contents and Skills Objectives

Topic 3.3 *(continued)*

3.4 General descriptive properties of conics

The student is able to:

- appreciate that the various conic sections (circle, ellipse, parabola, hyperbola and pairs of intersecting lines) are indeed the curves obtained when a plane intersects a (double) cone
- relate the various ranges of values of the eccentricity e to the appropriate conic and to understand how the shape of a conic varies as its eccentricity varies
- appreciate that the equations of all conic sections involve only quadratic expressions in x and y.

Topic 3: Conics

Applications, Implications and Considerations

Question 5. *(1988)*

The hyperbola H has equation $xy = 16$.

(a) Sketch this hyperbola and indicate on your diagram the positions and coordinates of all points at which the curve intersects the axes of symmetry.

(b) $P(4p, \frac{4}{p})$, where $p > 0$, and $Q(4q, \frac{4}{q})$, where $q > 0$, are two distinct arbitrary points on H. Find the equation of the chord PQ.

(c) Prove that the equation of the tangent at P is $x + p^2y = 8p$.

(d) The tangents at P and Q intersect at T. Find the coordinates of T.

(e) The chord PQ produced passes through the point $N(0, 8)$.

  (i) Find the equation of the locus of T.

  (ii) Give a geometrical description of this locus.

- No attempt should be made to derive equations using this approach. Its purpose is to provide a justification for the name ‘conic section’ and to indicate how the various shapes are all obtained as a single moving plane intersects a fixed cone. Use of a three-dimensional model, or of appropriate films, videos or graphics, is helpful.

- No formal treatment of this, or of the general quadratic equation in two variables is required. Students could be invited by the teacher to consider whether a curve specified by such an equation might always represent a conic section, using the general equation of a circle as an analogy.

Topic 4

Integration

Topic 4: Integration

Contents and Skills Objectives

4.1 Integration

The student is able to:

- use a table of standard integrals
- change an integrand into an appropriate form by use of algebra
- evaluate integrals using algebraic substitutions
- evaluate simple trigonometric integrals
- evaluate integrals using trigonometric substitutions
- evaluate integrals using integration by parts
- derive and use recurrence relations
- integrate rational functions by completing the square in a quadratic denominator
- integrate rational functions whose denominators have simple linear or quadratic factors.

Topic 4: Integration

Applications, Implications and Considerations

- Some of the results listed in the standard integrals table will need to be established as an appropriate method is developed.

- Some integrals may be changed into a form which can be integrated through use of some simple algebra, eg $\int\left(\frac{x+1}{x}\right)^2 dx$, $\int \frac{x^2}{x^2+1} dx$.

- Only simple substitutions are needed, eg $u = 1 + x^2$, $v^2 = 1 - x$ in $\int x(1+x^2)^4 dx$, $\int \frac{x}{\sqrt{1-x}} dx$. The effect on limits of integration is required, and definite integrals are to be treated.

- Include squares of all trigonometric functions, as well as those which can be found by a simple substitution, eg $\int_0^{\frac{\pi}{4}} \sin^2 2x\, dx$, $\int_0^{\frac{\pi}{4}} \sin^2 x \cos x\, dx$, $\int_0^{\frac{\pi}{4}} \sin^2 x \cos^3 x\, dx$.

- Typical substitutions would be $x = a \tan\theta$ and $t = \tan\frac{\theta}{2}$ in integrals such as $\int \frac{dx}{a^2+x^2}$, $\int \frac{\sin\theta}{2+\sin\theta}\, d\theta$.

- The work on integration by parts should include the integrands $\sin^{-1} x$, $e^{ax} \cos bx$, $\ln x$, $x^n \ln x$ (n an integer).

- Integration by parts should be extended to particular types of recurrence relations, eg $\int x^n e^x dx$, $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$. (Recurrence relations such as $\int_0^1 x^m(1-x)^n\, dx$, which involve more than one integer parameter, are excluded.)

- Examples should include cases to be integrated using a sum or difference of two squares, eg $\int \frac{dx}{x^2-4x-1}$, $\int \frac{dx}{3x^2+6x+10}$, $\int \frac{3x+2}{x^2-4x+1} dx$.

- Only rational functions, whose denominators can be broken into a product of distinct linear factors, or of a distinct quadratic factor and a linear factor, or of two distinct quadratic factors, need to be considered, eg
  $\int \frac{9x-2}{2x^2-7x+3}\, dx$, $\int \frac{3x^2-2x+1}{(x^2+1)(x^2+2)}\, dx$, $\int \frac{2x^2+3x-1}{x^3-x^2+x-1}\, dx$.

  Cases where the degree of the numerator is not less than the degree of the denominator are to be considered.

Topic 5

Volumes

Topic 5: Volumes

Contents and Skills Objectives

5.1 Volumes

The student is able to:

- appreciate that, by dividing a solid into a number of slices or shells, whose volumes can be simply estimated, the volume of the solid is the value of the definite integral obtained as the limit of the corresponding approximating sums

- find the volume of a solid of revolution by summing the volumes of slices with circular cross-sections

- find the volume of a solid of revolution by summing the volumes of slices with annular cross-sections

- find the volume of a solid of revolution by summing the volumes of cylindrical shells

- find the volume of a solid which has parallel cross-sections of similar shapes.

Topic 5: Volumes

Applications, Implications and Considerations

- The purpose of this topic is to provide practical examples of the use of a definite integral to represent a quantity (in this case, a volume) whose value can be regarded as the limit of an appropriate approximating sum. Emphasis is to be placed on understanding the various approximation methods given, deriving the relevant approximate expression for the corresponding element of volume and proceeding from this to expressing the volume as a definite integral. The evaluations of infinite series by a definite integral, or of integrals by summation of series, are not included in this topic.

- Volumes of revolution could lead, from questions involving rotation about a coordinate axis, to rotation about a line parallel to a coordinate axis, eg find the volume of the solid formed when the region bounded by $y = 2\sqrt{x}$, the x-axis and $x = 4$ is rotated about the line $x = 4$.

- Students should be encouraged to draw a sketch of the shape of the volume to be found and a sketch of a cross-sectional slice. They should then derive an expression for the volume of a cross-sectional slice in a form which leads directly to an expression for the total volume as an integral.

- Examples involving annular shells should include questions as difficult as the following. The region R, bounded by $0 \le x \le 2$, $0 \le y \le 4x^2 - x^4$, is rotated about the y-axis. The solid so formed is sliced by planes perpendicular to the y-axis. Express the areas of the cross-sections so formed as a function of y, the distance of the plane from the origin. Use this result to calculate the volume of the solid.

- A formula for summing by cylindrical shells should not be learnt. Each problem should rather be developed from first principles. An example of a more difficult problem is found in Question 5, 4 Unit HSC Mathematics paper, 1984 (see over).

- The process of writing the limiting sum as an integral should be extended to cases where cross-sections are other than circular. These cases should only involve problems in which the geometrical shape is able to be visualised, eg prove that the volume of a pyramid of height h on a square base of side a is $\frac{1}{3}a^2h$.

Topic 5. Volumes

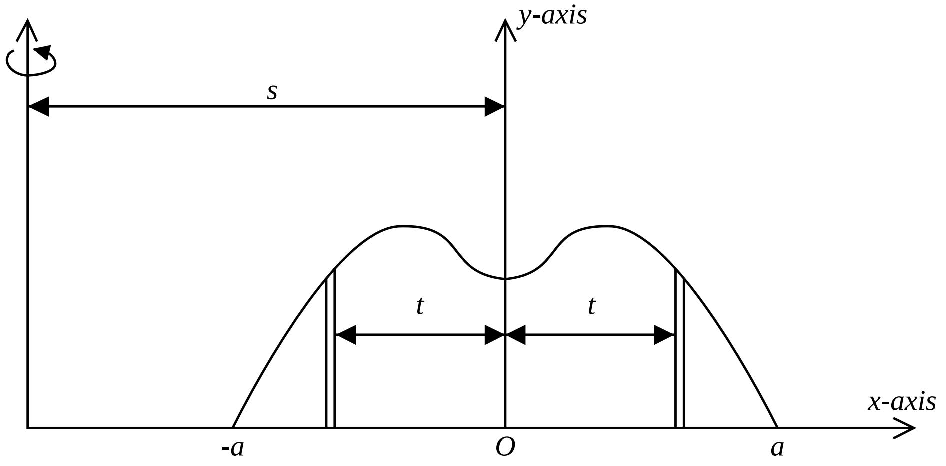
Contents and Skills Objectives

Topic 5.1 *(continued)*

Topic 5. Volumes

Applications, Implications and Considerations

Question 5. *(1984)*



(i) The diagram shows the area A between the smooth curve $y = f(x)$, $-a \le x \le a$, and the x-axis. (Note that $f(x) \ge O$ for $-a \le x \le a$ and $f(-a) = f(a) = O$.) The area A is rotated about the line $x = -s$ (where $s > a$) to generate the volume V. This volume is to be found by slicing A into thin vertical strips, rotating these to obtain cylindrical shells, and adding the shells. Two typical strips of width δt whose centre lines are distance t from the y-axis are shown.

(a) Show that the indicated strips generate shells of approximate volume $2\pi f(-t)\,(s - t)\delta t$, $2\pi f(t)(s + t)\delta t$, respectively.

(b) Assuming that the graph of f is symmetrical about the y-axis, show that $V = 2\pi sA$.

(ii) Assuming the results of part (i), solve the following problems.

(a) A doughnut shape is formed by rotating a circular disc of radius r about an axis in its own plane at a distance s ($s > r$) from the centre of the disc. Find the volume of the doughnut.

(b) The shape of a certain party jelly can be represented by rotating the area between the curve $y = \sin x$, $0 \le x \le \pi$, and the x-axis about the line $x = -\frac{\pi}{4}$. Find the volume generated.

Topic 6

Mechanics

Topic 6. Mechanics

Contents and Skills Objectives

6.1 Mathematical Representation of a motion described in physical terms

The student is able to:

- derive the equations of motion of a projectile
- use equations for horizontal and vertical components of velocity and displacement to answer harder problems on projectiles

- write down equations for displacement, velocity and acceleration given that a motion is simple harmonic
- use relevant formulae and graphs to solve harder problems on simple harmonic motion
- use Newton's laws to obtain equations of motion of a particle in situations other than projectile motion and simple harmonic motion
- describe mathematically the motion of particles in situations other than projectile motion and simple harmonic motion.

Topic 6. Mechanics

Applications, Implications and Considerations

- The aim of section 6.1 of the Mechanics topic is to provide students with the opportunity to do harder problems on the 3 Unit topic 'Applications of Calculus to the Physical World'.
- Students should be able to represent mathematically, motions described in physical terms. They should be able to explain, in physical terms, features given by mathematical descriptions of motion in one or two dimensions.
- The classical statement of Newton's First and Second laws of motion should be given as an illustration of the application of calculus to the physical world. Resolution of forces, accelerations and velocities in horizontal and vertical directions is to be used to obtain the appropriate equations of motion in two dimensions.
- Typical examples on projectile motion would include Question 6, 4 Unit HSC Mathematics paper, 1984 and Question 3, 4 Unit HSC Mathematics paper, 1979. (See p 59.)
- A typical example on simple harmonic motion would be: The deck of a ship was 2.4m below the level of a wharf at low tide and 0.6m above wharf level at high tide. Low tide was at 8.30 am and high tide at 2.35 pm. Find when the deck was level with the wharf, if the motion of the tide was simple harmonic.

- A typical such question would be Specimen Question 20, 4 Unit HSC Mathematics paper, 1981.

Question 20. *(1981)*

In this question assume that the earth is a sphere of radius R and that, at a point distant r ($\geq R$) from the centre of the earth, the acceleration due to gravity is proportional to r^{-2} and is directed towards the earth's centre; also, neglect forces due to all causes other than the earth's gravity.

A body is projected vertically upwards from the surface of the earth with initial speed V.

(i) Prove that it will escape from the earth (ie never return) if and only if $V \geq \sqrt{(2gR)}$ where g is the magnitude of the acceleration due to gravity at the earth's surface.

(ii) If $V = \sqrt{(2gR)}$ prove that the time taken to rise to a height R above the earth's surface is $\frac{1}{3}(4 - \sqrt{2})\sqrt{(R/g)}$.

Topic 6. Mechanics

Contents and Skills Objectives

Topic 6.1 *(continued)*

Topic 6. Mechanics

Applications, Implications and Considerations

Question 6. *(1984)*

Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection (upward inclination to the horizontal), α, but with different velocities U, V metres per second ($U < V$).

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(a) Show that, while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time. Hence, express the horizontal distance from P to the foot of the wall in terms of h, α.

(b) Show that

$$V(\tan \alpha + \tan \beta) = 2U \tan \alpha,$$

and deduce that, if $\beta = \frac{1}{2}\alpha$, then

$$U < \frac{3}{4}V.$$

Question 3. *(1979)*

Prove that the range on a horizontal plane of a particle projected upwards at an angle α to the plane with velocity V metres per second is $\frac{V^2 \sin 2\alpha}{g}$ metres, where g metres per second per second is the acceleration due to gravity.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second. The initial direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

Prove that, from a fixed position O on level ground, the sprinkler will wet the surface of an annular region with centre O and with internal and external radii $\frac{V^2}{2g}$ metres and $\frac{V^2}{g}$ metres respectively.

Deduce that, by locating the sprinkler appropriately relative to a rectangular garden bed of size 6 metres by 3 metres, the entire bed may be watered provided that $\frac{V^2}{2g} \geq 1 + \sqrt{7}$.

Topic 6. Mechanics

Contents and Skills Objectives

6.1.2 Physical explanations of mathematical descriptions of motion

The student is able to:

- given $\ddot{x} = f(x)$ and initial conditions derive $v^2 = g(x)$ and describe the resultant motion
- recognise that a motion is simple harmonic, given an equation for either acceleration, velocity or displacement, and describe the resultant motion.

Topic 6. Mechanics

Applications, Implications and Considerations

- A typical example requiring understanding of the mathematical description of a motion would be Question l0 (ii), 3 Unit HSC Mathematics paper, 1978.

Question 10(ii) *(1978)*

(ii) The motion of a pendulum may be approximately represented by the equation

$$\frac{d^2x}{dt^2} = -\frac{g}{L}x$$

where x is the displacement at time t. The constants g and L represent the value of gravity at the earth's surface and the length of the pendulum respectively. Consider the case $L = \frac{g}{\pi^2}$.

(a) Let $v = \frac{dx}{dt}$; show that $\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

(b) The pendulum is swinging in accordance with the initial conditions

$t = 0,\ x = 0,\ v = \pi.$

Using the approximate equation of motion above find the first value of x where $v = 0$.

(c) The motion of the pendulum may more accurately be represented by

$$\frac{d^2x}{dt^2} = -\frac{g}{L}\left[x - \frac{x^3}{6} + \frac{x^5}{120}\right].$$

Using this equation of motion and one application of Newton's method find a more accurate answer to (b).

Topic 6. Mechanics

Contents and Skills Objectives

6.2 Resisted motion

6.2.1 Resisted Motion along a horizontal line

The student is able to:

- derive, from Newton's laws of motion, the equation of motion of a particle moving in a single direction under a resistance proportional to a power of the speed
- derive an expression for velocity as a function of time (where possible)
- derive an expression for velocity as a function of displacement (where possible)
- derive an expression for displacement as a function of time (where possible).

6.2.2 Motion of a particle moving upwards in a resisting medium and under the influence of gravity

The student is able to:

- derive, from Newton's laws of motion, the equation of motion of a particle, moving vertically upwards in a medium, with a resistance R proportional to the first or second power of its speed
- derive expressions for velocity as a function of time and for velocity as a function of displacement (or vice versa)
- derive an expression for displacement as a function of time
- solve problems by using the expressions derived for acceleration, velocity and displacement.

Topic 6. Mechanics

Applications, Implications and Considerations

- Typical cases to consider include those in which the resistance is proportional to the speed and to the square of the speed.
- Analysis of the motion of a particle should include consideration of the behaviour of the particle as t becomes large. Graphs offer assistance in understanding the behaviour of the particle.
- Problems should be discussed up to the standard of Question 6, 4 Unit HSC Mathematics paper, 1987.

Question 6. *(1987)*

A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(a) Show that v is related to the displacement x by the formula

$$x = \tan^{-1}\left[\frac{Q - v}{1 + Qv}\right].$$

(b) Show that the time t which has elapsed when the particle is travelling with velocity v is given by

$$t = \frac{1}{2}\log_e\left[\frac{Q^2(1 + v^2)}{v^2(1 + Q^2)}\right].$$

(c) Find v^2 as a function of t.

(d) Find the limiting values of v and x as $t \to \infty$.

- Cases, other than where the resistance is proportional to the first or second power of the speed, are not required to be investigated.
- Students should be advised to place the origin at the point of projection.
- The maximum height reached by the particle can be obtained from the expression relating speed and displacement.
- The time taken to reach this maximum height can be obtained from the expression relating speed and time.
- Problems should include cases where the magnitude of the resistance is given (eg $R = \frac{1}{10}v^2$).

Topic 6. Mechanics

Contents and Skills Objectives

6.2.3 Motion of a particle falling downwards in a resisting medium and under the influence of gravity

The student is able to:

- derive, from Newton's laws of motion, the equation of motion of a particle falling in a medium with a resistance R proportional to the first or second power of its speed
- determine the terminal velocity of a falling particle, from its equation of motion
- derive expressions for velocity as a function of time and for velocity as a function of displacement
- derive an expression for displacement as a function of time
- solve problems by using the expressions derived for acceleration, velocity and displacement.

Topic 6. Mechanics

Applications, Implications and Considerations

- Cases, other than where the resistance is proportional to the first or second power of the speed, are not required to be investigated.

- Students should place the origin at the point from which the particle initially falls. If the motion of a particle both upwards and then downwards is considered then the position of the origin should be changed as soon as the particle reaches its maximum height. Care must then be taken in determining the correct initial conditions for the downward motion.

- The terminal velocity can be calculated from the equation of motion by finding V when $\ddot{x} = 0$.

- The time taken for the particle to reach ground level should be found.

- Problems should include a study of the complete motion of a particle, projected vertically upwards, which then returns to its starting point. For specific resistance functions, comparisons should be made between the times required for its upward and downward journeys and between the speed of projection and the speed of its return.

Topic 6. Mechanics

Contents and Skills Objectives

6.3 Circular Motion

6.3.1 Motion of a particle around a circle

The student is able to:

- define the angular velocity of a point moving about a fixed point
- deduce, from this definition of angular velocity, expressions for angular acceleration of a point about a fixed point
- prove that the instantaneous velocity of a particle moving in a circle of radius R, with angular velocity ω, is $R\omega$
- prove that the tangential and normal components of the force acting on a particle moving in a circle of radius R, with angular velocity ω, need to be $Mr\dot{\omega}$ and $-mR\omega^2$ respectively.

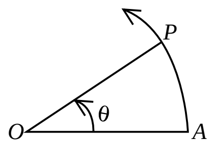
6.3.2 Motion of a particle moving with uniform angular velocity around a circle

The student is able to:

- write down the formulae appropriate for a particle moving around a circle with uniform angular velocity
- apply these formulae to the solution of simple problems.

Topic 6. Mechanics

Applications, Implications and Considerations

- 

A particle P moves about O along a curve AP. OA is a fixed arbitrary line about O. At time t, $A\hat{O}P = \theta$. Then the angular velocity of P about O is defined as $\omega = \frac{d\theta}{dt}$. The units for angular velocity must be radians per second.

- Differentiating this, results for angular acceleration $\left(\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = \frac{d}{d\theta}(\tfrac{1}{2}\omega^2)\right)$ follow easily.
- Taking displacement $x = R\theta$, the result $V = R\omega$ follows by differentiating with respect to t.

- The components are obtained by putting $x = R\cos\theta$ and $y = R\sin\theta$, where $P(x,y)$ is a point of a circle of radius R. Then, on differentiating, $\frac{dx}{dt} = -R\sin\theta.\dot{\theta}$ and $\frac{dy}{dt} = R\cos\theta.\dot{\theta}$. On differentiating again, $\ddot{x} = -R\sin\theta.\ddot{\theta} - R\cos\theta.(\dot{\theta})^2$ and $\ddot{y} = -R\cos\theta.\ddot{\theta} - R\sin\theta.(\dot{\theta})^2$.

  Hence $F_x = -mR\sin\theta.\ddot{\theta} - mR\cos\theta.(\dot{\theta})^2$, $F_y = mR\cos\theta.\ddot{\theta} - mR\sin\theta.(\dot{\theta})^2$.
  Resolving these forces in the direction of the tangent and normal leads to $F_N = -mR\omega^2$ and $F_T = mR\dot{\omega}$.

- Typical exercises could include:
  - the angular speed of a wheel attached to a child's electric motor is 30 revolutions per minute. The diameter of the wheel is 2 cm. Calculate the speed of a mark on the edge of the wheel.
  - a string is 50 cm long and will break if a mass exceeding 40 kilograms is hung from it. A mass of 2 kilograms is attached to one end of the string and it is revolved in a circle upon a smooth horizontal table. Find the greatest angular velocity which may be imparted without breaking the string.

Topic 6. Mechanics

Contents and Skills Objectives

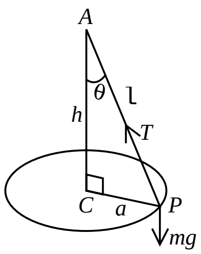
6.3.3 The Conical Pendulum

The student is able to:

- use Newton's law to analyse the forces acting on the bob of a conical pendulum
- derive results including $\tan\theta = \frac{v^2}{ag}$ and $h = \frac{g}{\omega^2}$
- discuss the behaviour of the pendulum as its features vary
- apply derived formulae to the solution of simple problems.

Topic 6. Mechanics

Applications, Implications and Considerations

- 

  In a vertical direction, $T \cos \theta - mg = 0$.

  Now the force acting towards the centre is of magnitude $T \sin\theta$.

  $\therefore\ T \sin\theta = \dfrac{mv^2}{a} = ma\omega.$

  $\therefore\ \tan\theta = \dfrac{v^2}{ag} = \dfrac{a\omega^2}{g}.$

  But in ΔACP, $\tan\theta = \frac{a}{h}$. $\therefore\ h = g/\omega^2$.

- Features, that should be noted, would include the facts that the vertical depth of the bob below A is independent of the length of the string and the mass of the bob, and that, as the speed of the particle increases, the particle rises.

- Discussion of the behaviour of the pendulum should be accompanied by basic practical experiences.

- Simple problems include those of the following type: the number of revolutions per minute of a conical pendulum increases from 60 to 90. Find the rise in the level of the bob.

Topic 6. Mechanics

Contents and Skills Objectives

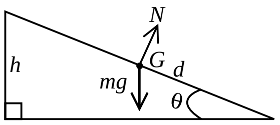
6.3.4 Motion around a banked circular track

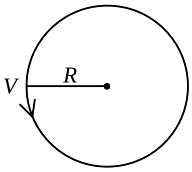
The student is able to:

- Use Newton's laws to analyse the forces acting on a body, represented by a particle, moving at constant speed around a banked circular track
- derive the results $\tan\theta = \frac{v^2}{Rg}$ and $h = \frac{v^2 d}{Rg}$
- calculate the optimum speed around a banked track given the construction specifications
- calculate the forces acting on a body, travelling around a banked track, at a speed other than the optimum speed.

Topic 6. Mechanics

Applications, Implications and Considerations





- A car, considered as a point mass G, is travelling at constant speed v around a curve of radius R on a road of width d. The road slopes at angle θ towards the centre of the curve. If there is no lateral force on the wheels of the car, find h, the difference in height across the width of the road.

- Since no lateral force is exerted by the road on the car, the reaction N is normal to the road surface, as shown. Using the other given information and resolving horizontally and vertically

$$N\sin\theta = mR\omega^2 = \frac{mv^2}{R} \text{ and } N\cos\theta = mg.$$

$$\therefore\ \tan\theta = \frac{v^2}{Rg}. \text{ For small } \theta,\ \sin\theta = \frac{h}{d} \approx \tan\theta.$$

$$\text{Thus } h = \frac{v^2 d}{Rg}.$$

Typical exercises could include:

- a railway line has been constructed around a circular curve of radius 400 metres. The distance between the rails is 1.5 metres. The outside rail is 0.08 metres above the inside rail. Find the most favourable speed (the speed which eliminates a sideways force on the wheels) for a train on this curve.

- a car travels at v metres/second along a curved track of radius R metres. Find the inclination of the track to the horizontal, if there is to be no tendency for the car to slip sideways. If the speed of a second car, of mass M kilograms, is V metres/second, prove that the sideways frictional force, exerted by the surface on the wheels of this car, is

$$\frac{Mg(V^2 - v^2)}{\sqrt{V^4 + R^2 g^2}}.$$

Topic 7

Polynomials

Topic 7. Polynomials

Contents and Skills Objectives

7.1 Integer roots of polynomials with integer coefficients

The student is able to:

- prove that, if a polynomial has integer coefficients and if a is an integer root, then a is a divisor of the constant term
- test a given polynomial with integer coefficients for possible integer roots

7.2 Multiple Roots

The student is able to:

- define a multiple root of a polynomial
- write down the order (multiplicity) of a root
- prove that if $P(x) = (x - a)^r S(x)$, where $r > 1$ and $S(a) \neq 0$, then $P'(x)$ has a root a of multiplicity $(r - 1)$
- solve simple problems involving multiple roots of a polynomial.

7.3 Fundamental Theorem of Algebra

The student is able to:

- state the fundamental theorem of algebra
- deduce that a polynomial of degree $n > 0$, with real or complex coefficients, has exactly n complex roots, allowing for multiplicities.

Topic 7. Polynomials

Applications, Implications and Considerations

- All possible integer roots of such a polynomial therefore lie among the positive and negative integer divisors of its constant term.

- It may happen that $P(x) = (x - a)Q(x)$ and that $(x - a)$ is a factor of $Q(x)$. The number a is then called a repeated or multiple root of $P(x)$.
- If $P(x) = (x - a)^r S(x)$, where r is a positive integer and $S(a) \neq 0$. then a is a root of $P(x)$ of order (or multiplicity) r. $(x - a)$ is called a factor of $P(x)$ of order r. A simple root corresponds to a factor of order 1.
- A typical problem on multiple roots is Question 3(ii), 4 Unit HSC Mathematics paper, 1986.

Question 3(ii) *(1986)*

(a) Show that if a is a multiple root of the polynomial equation $f(x) = 0$ then $f(a) = f'(a) = 0$.

(b) The polynomial

$$\alpha x^{n+1} + ßx^n + 1$$

is divisible by $(x - 1)^2$. Show that $\alpha = n$, and $ß = -(1 + n)$.

(c) Prove that $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ has no multiple roots for any $n \geq 1$.

- The 'fundamental theorem of algebra' asserts that every polynomial $P(x)$ of degree n over the complex numbers has at least one root.
- Using this result, the factor theorem should now be used to prove (by induction on the degree) that a polynomial of degree $n > 0$ with real (or complex) coefficients has exactly n complex roots (each counted according to its multiplicity) and is expressible as a product of exactly n complex linear factors.

Topic 7. Polynomials

Contents and Skills Objectives

7.4 Factoring Polynomials

The student is able to:

- recognise that a complex polynomial of degree *n* can be written as a product of *n* complex linear factors
- recognise that a real polynomial of degree *n* can be written as a product of real linear and real quadratic factors
- factor a real polynomial into a product of real linear and real quadratic factors
- factor a polynomial into a product of complex linear factors
- write down a polynomial given a set of properties sufficient to define it
- solve polynomial equations over the real and complex numbers.

7.5 Roots and Coefficients of a Polynomial Equation

The student is able to:

- write down the relationships between the roots and coefficients of polynomial equations of degrees 2, 3 and 4.
- use these relationships to form a polynomial equation given its roots
- form an equation, whose roots are a multiple of the roots of a given equation
- form an equation, whose roots are the reciprocals of the roots of a given equation
- form an equation, whose roots differ by a constant from the roots of a given equation
- form an equation, whose roots are the squares of the roots of a given equation.

Topic 7. Polynomials

Applications, Implications and Considerations

- The fact that complex roots of real polynomials occur in conjugate pairs leads directly to the factorisation of real polynomials over the real numbers as a product of real linear and real quadratic factors. In particular, a real polynomial of odd degree always has at least one real root.
- Students should be able to factor cubic and quartic polynomials over both the real and complex numbers.
- Students should be able to factor polynomials with a degree greater than 4 in cases where factors are possible to obtain by other than the remainder theorem (eg $x^6 - 1$, $z^5 + 16z$).

- A typical question on the relationship between roots and coefficients of a polynomial was Question 7(ii), 4 Unit HSC Mathematics paper, 1982.

Question 7(ii) *(1982)*

- Use De Moivre's theorem to express $\cos 5\theta$, $\sin 5\theta$ in powers of $\sin\theta$ and $\cos\theta$. Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$.

  Deduce that:

  $\tan\frac{\pi}{5}\ \tan\frac{2\pi}{5}\ \tan\frac{3\pi}{5}\ \tan\frac{4\pi}{5} = 5.$

- The simplest approach to forming an equation whose roots are related to the roots of a given equation often does not involve using the relationship between the roots and coefficients. An equation, whose roots are m times those of a polynomial equation $P(x) = 0$, is $P(x/m) = 0$. For example, a cubic equation $ax^3 + bx^2 + cx + d = 0$ may have roots α, $ß$, γ. Then an equation with roots $m\alpha$, $mß$, $m\gamma$ is $a(x/m)^3 + b(x/m)^2 + c(x/m) + d = 0$. An equation, whose roots are reciprocals of those of $P(x) = 0$, is $P(1/x) = 0$. Thus a cubic whose roots are $1/\alpha$, $1/ß$, $1/\gamma$ is $a/x^3 + b/x^2 + c/x + d = 0$ or $a + bx + cx^2 + dx^3 = 0$. An equation, whose roots are all k less than those of $P(x) = 0$, is $P(x + k) = 0$. Thus, a cubic with roots $\alpha - k$, $ß - k$, $\gamma - k$ is $a(x + k)^3 + b(x + k)^2 + c(x + k) + d = 0$.
  An equation, whose roots are the squares of those of $P(x) = 0$, is $P(\sqrt{x}) = 0$ (converted to a polynomial in x). Thus, an equation with roots α^2, $ß^2$, γ^2 is $\sqrt{x}\,(ax + c) + bx + d = 0$, or $x(ax + c)^2 = (bx + d)^2$.

Topic 7. Polynomials

Contents and Skills Objectives

Topic 7.5 *(continued)*

7.6 Partial Fractions

The student is able to:

- write $f(x) = \frac{A(x)}{B(x)}$, where deg $A(x) \geq$ deg $B(x)$, in the form $f(x) = Q(x) + R(x)/B(x)$, where deg $R(x) <$ deg $B(x)$
- write $\frac{R(x)}{B(x)}$, where deg $R(x) <$ deg $B(x)$ and $B(x)$ is a product of distinct linear factors $c(x - a_1) \dots (x - a_n)$, in the form $\frac{c_1}{x - a_1} + \dots + \frac{c_n}{x - a_n}$
- write $\frac{R(x)}{B(x)}$, where deg $R(x) <$ deg $B(x)$ and $B(x)$ is a product of distinct linear factors and a simple quadratic factor, in the form $\frac{c_1}{x - a_1} + \dots + \frac{c_n}{x - a_n} + \frac{dx + e}{x^2 + bx + c}$
- write $\frac{R(x)}{B(x)}$, where deg $R(x) <$ deg $B(x)$ and $B(x)$ is a product of two different simple quadratic factors of form $x^2 + b_i$, in the form $\frac{c_1 x + d_1}{x^2 + b_1} + \frac{c_2 x + d_2}{x^2 + b_2}$.
- apply these partial fraction decompositions to the integration of corresponding functions.

Topic 7. Polynomials

Applications, Implications and Considerations

- A typical question on formation of an equation with related roots is Question 7(ii), 4 Unit HSC Mathematics paper, 1983.

Question 7(ii) *(1983)*

Let α, ß, γ be the roots of the equation

$x^3 + qx + r = 0$,

where $r \neq 0$. Obtain as functions of q, r, in their simplest forms, the coefficients of the cubic equations whose roots are:

(a) $\dot{\alpha}^2$, $ß^2$, γ^2;

(b) α^{-1}, $ß^{-1}$, γ^{-1};

(c) α^{-2}, $ß^{-2}$, γ^{-2}.

- If deg $A(x) \geq$ deg $B(x)$, then $A(x)$ must be divided initially by $B(x)$ in order to write $\frac{A(x)}{B(x)}$ into a form upon which a partial fraction decomposition may be applied.

- A variety of methods should be examined, in carrying out a decomposition of $\frac{R(x)}{B(x)}$ into partial fractions, when $B(x)$ is a product of distinct linear factors.

  If $\frac{R(x)}{B(x)} = \frac{c_1}{x - a_1} + \ldots + \frac{c_n}{x - a_n}$, then

  $R(x) = c_1(x - a_2)(x - a_3)\ldots(x - a_n) + \ldots + c_n(x - a_1)(x - a_2)\ldots(x - a_{n-1})$. Coefficients may then be obtained by equating coefficients, substituting in carefully selected values of x or using the fact that $c_i = R(a_i)/B'(a_i)$. This may be derived by noting that, if $\frac{R(x)}{B(x)} = \frac{c_1}{x - a_1} + \ldots + \frac{c_n}{x - a_n}$ and $B(a_1) = 0$, then

  $$R(x) \cdot \frac{x - a_1}{B(x) - B(a_1)} = c_1 + (x - a_1)\frac{c_2}{x - a_2} + \ldots + (x - a_1)\frac{c_n}{x - a_n}.$$

  Let x tend to a_1. Then LHS $\to R(a_1)/B'(a_1)$ and RHS $\to c_1$.

  $\therefore \quad c_1 = R(a_1)/B'(a_1)$.

- Cases when multiple factors of $B(x)$ exist (eg $B(x) = (x - a_1)^n D(x)$) are not included in this course.

Topic 8

Harder 3 Unit Topics

Topic 8. Harder 3 Unit

Contents and Skills Objectives

The whole of the syllabus for the 3 Unit course is included in the 4 Unit course. During their 4 Unit course, students should gain experience with harder problems on all 3 Unit topics and it is anticipated that such problems would be included in any 4 Unit examination. The level of difficulty of illustrative examples in the notes in the 3 Unit syllabus is not intended to limit the difficulty of questions which could be set for students. The following treatment of topics is intended to indicate the manner in which each topic could be developed with 4 Unit students.

8.1 Geometry of the Circle

The student is able to:

- solve more difficult problems in geometry.

8.2 Induction

The student is able to:

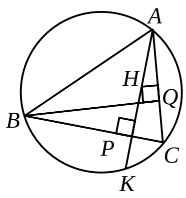
- carry out proofs by mathematical induction in which $S(1)$, $S(2) \ldots S(k)$ are assumed to be true in order to prove $S(k + 1)$ is true
- use mathematical induction to prove results in topics which include geometry, inequalities, sequences and series, calculus and algebra.

Topic 8. Harder 3 Unit

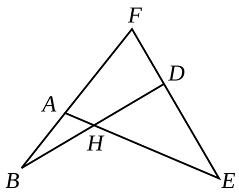
Applications, Implications and Considerations

- Students should have experience in solving more difficult geometry problems than those indicated in the 3 Unit syllabus. Proofs could involve the use of any of the listed results.
- The following examples on concyclic points illustrate the depth of treatment that 4 Unit students should encounter with all results in the course on deductive geometry.
  - The altitudes *AP* and *BQ* in an acute-angled triangle *ABC* meet at *H*. *AP* produced cuts the circle through *A*, *B* and *C* at *K*. Prove that

    $$HP = PK.$$



  - *BF*, *FE*, *AE* and *BD* are 4 straight lines as in the diagram. *AE* meets *BD* at *H*. Circles are drawn through the vertices of the four triangles *ABH*, *HDE*, *FBD* and *FAE*. Prove that the four circles have a common point.



- At times, students may need to assume $S(1)$, $S(2) \ldots S(k)$ are true in order to prove $S(k + 1)$ is true.
  - Prove that the angle sum of an *n* sided figure is equal to $2n - 4$ right angles.
  - A sequence $\{u_n\}$ is such that $u_{n+3} = 6u_{n+2} - 5u_{n+1}$, and $u_1 = 2$, $u_2 = 6$. Prove that $u_n = 5^{n-1} + 1$.
- In the 4 Unit course, students should see proofs by mathematical induction in a variety of topics. Questions, in which the result proven is subsequently used, should be given to students. Examples which illustrate this idea include Question 8(ii), 4 Unit HSC Mathematics paper, 1985 and Question 8(ii), 4 Unit HSC Mathematics paper, 1981.

Topic 8. Harder 3 Unit

Contents and Skills Objectives

8.2 *(continued)*

Topic 8. Harder 3 Unit

Applications, Implications and Considerations

Question 8(ii) *(1985)*

(a) Show that for $k \geq 0$,

$$2k + 3 > 2\sqrt{\{(k+1)(k+2)\}}.$$

(b) Hence prove that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2[\sqrt{(n+1)} - 1]$$

(c) Is the statement that, for all positive integers N,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10},$$

true? Give reasons for your answer.

Question 8(ii) *(1981)*

Using induction, show that for each positive integer n there are unique positive integers p_n and q_n such that

$$(1+\sqrt{2})^n = p_n + q_n\sqrt{2}.$$

Show also that $p_n^2 - 2q_n^2 = (-1)^n$.

Topic 8. Harder 3 Unit

Contents and Skills Objectives

8.3 Inequalities

The student is able to:

- prove simple inequalities by use of the definition of $a > b$ for real a and b
- prove further results involving inequalities by logical use of previously obtained inequalities.

Topic 8. Harder 3 Unit

Applications, Implications and Considerations

- Simple questions, which depend on the concept that $a > b$ if and only if $(a - b) > 0$, should be done by students. Eg prove that, when x, y, z are real and not all equal, $x^2 + y^2 + z^2 > yz + zx + xy$, and deduce that, if also $x + y + z = 1$, then $yz + zx + xy < 1/3$.
- Establishing the relationship between the arithmetic and geometric mean often leads to an elegant solution; eg

(i) Prove that $\frac{a+b}{2} \geq \sqrt{ab}$ if a and b are positive real numbers.

(ii) Given that $x + y = p$, prove that, if $x > 0$, $y > 0$, then $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{p}$ and $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{8}{p^2}$.

Solution:

(i) $\left(\frac{a+b}{2}\right)^2 - ab = \frac{1}{4}\,[a^2 + b^2 + 2ab - 4ab] = \frac{1}{4}\,(a - b)^2 \geq 0.$

$\therefore\ \frac{a+b}{2} \geq \sqrt{ab}$, for positive real numbers a, b.

(ii) Using (i) with $a = \frac{1}{x}$, $b = \frac{1}{y}$,

$$\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}$$

Now, using (i) with $a = x$, $b = y$,

$$\sqrt{xy} \leq \frac{x+y}{2} = \frac{p}{2}.$$

Hence $\frac{1}{x} + \frac{1}{y} \geq 2 \cdot \frac{1}{\sqrt{xy}} \geq \frac{4}{p}$

Using (i) with $a = \frac{1}{x^2}$, $b = \frac{1}{y^2}$,

$$\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{2}{\sqrt{x^2y^2}} = \frac{2}{xy}$$

Since $\sqrt{xy} \leq \frac{p}{2}$, $xy \leq \frac{p^2}{4}$

Hence $\frac{1}{x^2} + \frac{1}{y^2} \geq 2 \cdot \frac{1}{xy} \geq \frac{8}{p^2}$