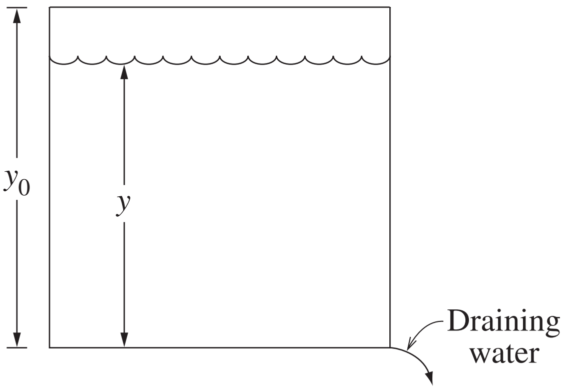
Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram represents a vertical cylindrical water cooler of constant cross-sectional area A. Water drains through a hole at the bottom of the cooler. From physical principles, it is known that the volume V of water decreases at a rate given by

$$\frac{dV}{dt} = -k\sqrt{y} ,$$

where k is a positive constant and y is the depth of water.

Initially the cooler is full and it takes T seconds to drain. Thus $y = y_0$ when $t = 0$, and $y = 0$ when $t = T$.

(i) Show that $\dfrac{dy}{dt} = -\dfrac{k}{A}\sqrt{y}$. **1**

(ii) By considering the equation for $\dfrac{dt}{dy}$, or otherwise, show that **4**

$$y = y_0\left(1 - \frac{t}{T}\right)^2 \quad \text{for } 0 \le t \le T.$$

(iii) Suppose it takes 10 seconds for half the water to drain out. How long does it take to empty the full cooler? **2**

Question 7 continues on page 12

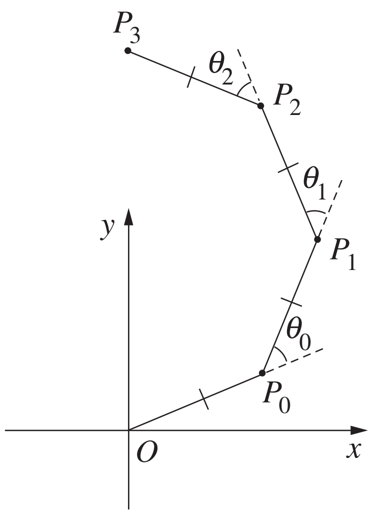
Marks

Question 7 (continued)

(b) Suppose $0 < \alpha,\ \beta < \dfrac{\pi}{2}$ and define complex numbers z_n by

$$z_n = \cos(\alpha + n\beta) + i\sin(\alpha + n\beta)$$

for $n = 0, 1, 2, 3, 4$. The points P_0, P_1, P_2 and P_3 are the points in the Argand diagram that correspond to the complex numbers z_0, $z_0 + z_1$, $z_0 + z_1 + z_2$ and $z_0 + z_1 + z_2 + z_3$ respectively. The angles θ_0, θ_1 and θ_2 are the external angles at P_0, P_1 and P_2 as shown in the diagram below.



(i) Using vector addition, explain why **2**

$$\theta_0 = \theta_1 = \theta_2 = \beta.$$

(ii) Show that $\angle P_0OP_1 = \angle P_0P_2P_1$, and explain why $OP_0P_1P_2$ is a cyclic quadrilateral. **2**

(iii) Show that $P_0P_1P_2P_3$ is a cyclic quadrilateral, and explain why the points O, P_0, P_1, P_2 and P_3 are concyclic. **2**

(iv) Suppose that $z_0 + z_1 + z_2 + z_3 + z_4 = 0$. Show that **2**

$$\beta = \frac{2\pi}{5}.$$

End of Question 7

7/ a)

$$\frac{dV}{dt} = -k\sqrt{y}$$

$$\frac{dy}{dt} = \frac{dy}{dV} \cdot \frac{dV}{dt}$$

$$V = yA$$

$$\therefore \frac{dV}{dy} = A$$

$$\frac{dy}{dV} = \frac{1}{A}$$

$$\therefore \frac{dy}{dt} = \frac{1}{A} \cdot -k\sqrt{y}$$

$$= -\frac{k}{A}\sqrt{y}$$

$$\frac{dy}{\sqrt{y}} = -\frac{k\,dt}{A}$$

$$\therefore 2\sqrt{y} = -\frac{k}{A}t + c$$

$t = 0$, $y = y_0$

$$\therefore c = 2\sqrt{y_0}$$

$t = T$, $y = 0$

$$\therefore \frac{kT}{A} = 2\sqrt{y_0}$$

$$\therefore k = \frac{2A}{T}\sqrt{y_0}$$

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$$\therefore \quad 2\sqrt{y} = -\frac{k}{A}t + 2\sqrt{y_0}$$

$$= -\frac{t}{\cancel{A}} \cdot \frac{2\cancel{A}}{T}\sqrt{y_0} + 2\sqrt{y_0}$$

$$= -\frac{2t}{T}\sqrt{y_0} + 2\sqrt{y_0}$$

$$\cancel{2}\sqrt{y} = \cancel{2}\sqrt{y_0}\left(1 - \frac{t}{T}\right)$$

$$y = y_0\left(1 - \frac{t}{T}\right)^2$$

for $0 \le t \le T$

$t = 10, \quad y = \frac{1}{2}y_0$

$$\frac{1}{2}\cancel{y_0} = \cancel{y_0}\left(1 - \frac{10}{T}\right)^2$$

$$\therefore \quad 1 - \frac{10}{T} = \pm\frac{1}{\sqrt{2}}$$

$$\frac{10}{T} = 1 \pm \frac{1}{\sqrt{2}}$$

$$T = \frac{10}{1 \pm \frac{1}{\sqrt{2}}}$$

~~If $1 + \frac{1}{\sqrt{2}}$, $T = \frac{10}{1+\frac{1}{\sqrt{2}}}$~~ $5.85 < 10$

not possible

~~$\therefore T = \frac{10}{1 - \frac{1}{\sqrt{2}}} = 34$ sec~~

we required $T > 10$

$\therefore \dfrac{10}{x} > 10$ where $x = 1 \pm \dfrac{1}{\sqrt{2}}$

$x < 1$

$\therefore x = 1 - \dfrac{1}{\sqrt{2}}$ not $1 + \dfrac{1}{\sqrt{2}}$

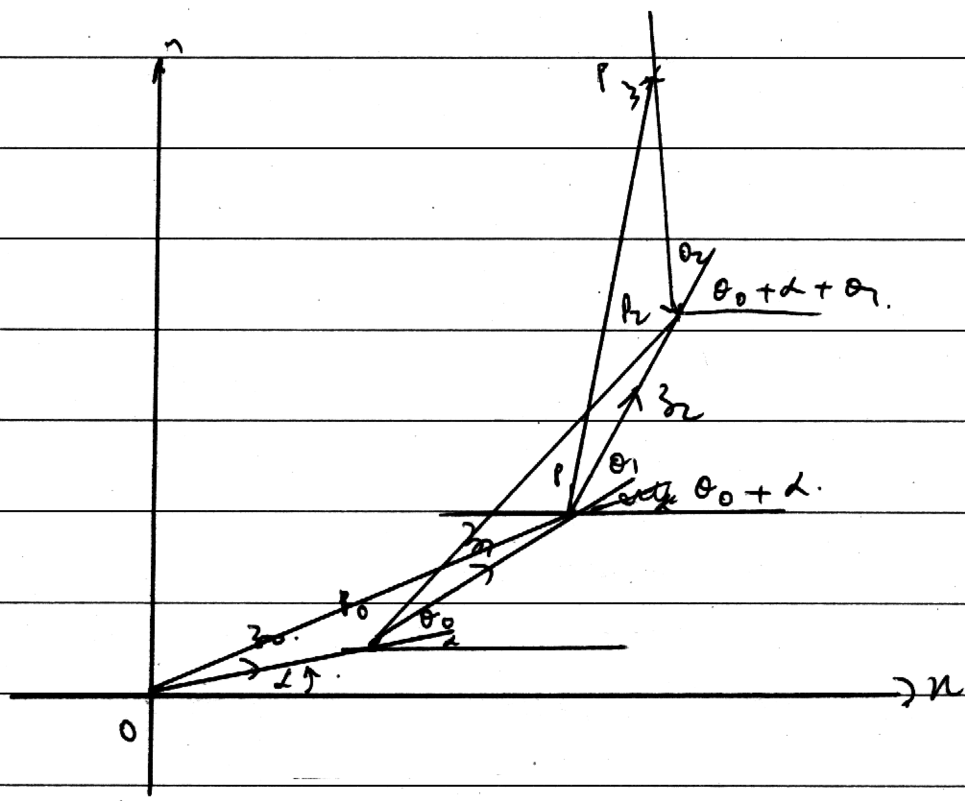
$\therefore T = \dfrac{10}{1 - \dfrac{1}{\sqrt{2}}}$

$\doteqdot$ 34.14 sec

7(iv) $0 < \alpha, \beta, < \frac{\pi}{2}$

$z_n = \cos(\alpha + n\beta) + i \sin(\alpha + n\beta)$

$n = 0, 1, 2, 3, 4$



$P_0, \quad z_0 = \cos\alpha + i\sin\alpha$

$P_1 \Rightarrow z_0 + z_1$ $\qquad \overrightarrow{P_0P_1} = z_1 + z_0 - z_0$

$P_2 \to z_0 + z_1 + z_2$. $\qquad = z_1$.

c) $\overrightarrow{OP_1} + z_2$.

$\overrightarrow{P_1P_2} = z_0 + z_1 + z_2 - z_0 - z_1$

$= z_2$.

arg P_1P_2 = arg z_2

$= \alpha + 2\beta$ from diagram,

$\cancel{\alpha} + 2\beta = \theta_0 + \theta_1 + \cancel{\alpha}$

$\theta_0 + \theta_1 = 2\beta$

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$\arg \overrightarrow{P_0P_1} = \arg z_1$

$= \alpha + \beta$

$= \alpha + \theta_1$

$\therefore \beta = \theta_1$

$\therefore \theta_0 = \theta_1$

$\arg \overrightarrow{P_2P_3} = \arg z_3$

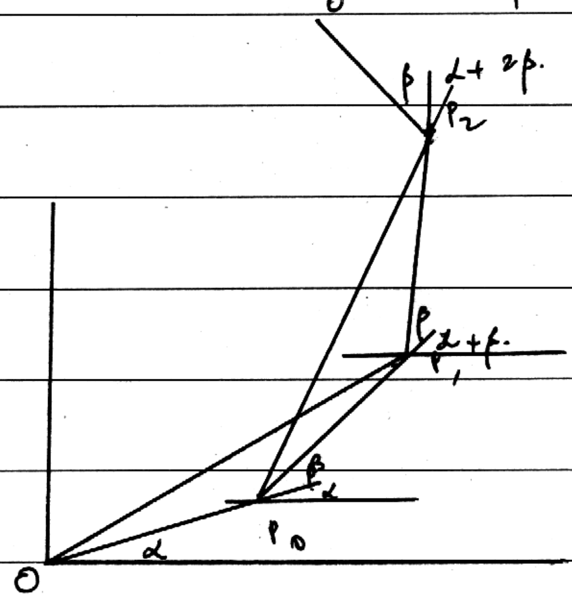
$= \alpha + 3\beta = \theta_0 + \alpha + \theta_1 + \theta_2$

$= \alpha + 2\beta + \theta_2$

$\therefore \theta_2 = \beta$

~~$\theta_0 = \theta_2 = \theta_3$~~

$\theta_0 = \theta_1 = \theta_2 = \beta$



In ΔOP_0P_1 and $\Delta P_0P_1P_2$

$|OP_0| = |P_0P_1|$

since $|z_0| = |z_1|$

and $|P_0P_1| = |P_1P_2|$

since $z_1 = |z_2|$

$\angle P_2 P_1 P_0 = 180 - \beta$. (~~adj~~ adjacent angle on straight line are supplementary)

$\angle P_1 P_0 O = 180^\circ - \beta$ (adjacent angles on straight line are supplementary)

$\therefore \angle P_2 P_1 P_0 = \angle P_1 P_0 O$

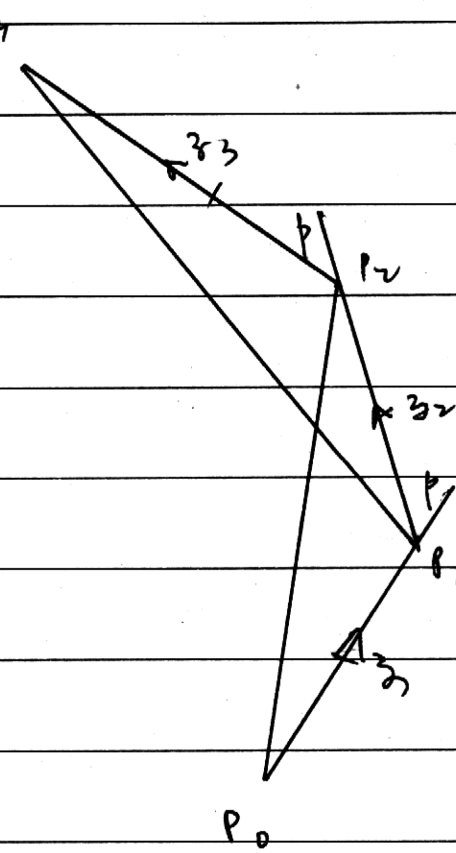
$\therefore \triangle O P_0 P_1 \equiv \triangle P_0 P_1 P_2$ (SAS)

$\therefore$ ~~$\angle P_1 OP_0 = \angle$~~

$\therefore \angle P_1 O P_0 = \angle P_0 P_2 P_1$ (corresponding angles of congruent triangles $\triangle OP_0P_1 \equiv \triangle P_0 P_1 P_2$ equal)

since $\angle P_1 O P_0 = \angle P_0 P_2 P_1$, then

$O P_0 P_1 P_2$ is cyclic, since of a line $P_0 P_1$ subtends equal angles at two points on the same side of it, then the four points are concyclic



In $\Delta P_0 P_1 P_2$, $\Delta P_1 P_2 P_3$

$|P_0 P_1| = |P_1 P_2|$

since $|z_1| = |z_2|$

$|P_1 P_2| = |P_3 P_2|$

since $|z_2| = |z_3|$

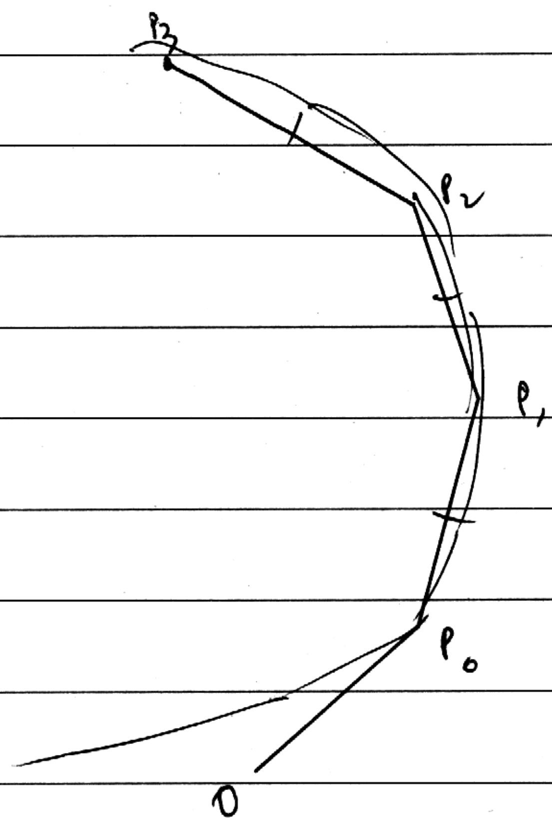
$\angle P_3 P_2 P_1 = \angle P_2 P_1 P_0$

$= 180 - \beta$

$\therefore \Delta P_0 P_1 P_2 \equiv \Delta P_1 P_2 P_3$ (SAS)

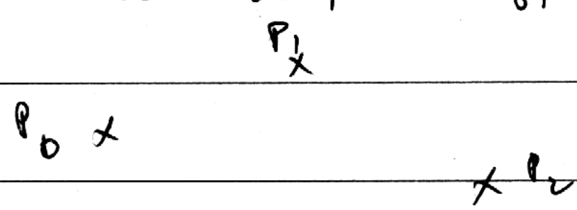
$\therefore \angle P_2 P_3 P_1 = \angle P_1 P_0 P_2$ (corresponding ~~sides~~ angle of congruent triangles $\Delta P_0 P_1 P_2 \equiv \Delta P_1 P_2 P_3$ equal)

$\therefore P_0P_1P_2P_3$ is a cyclic quad, since if a line P_1P_2 subtends equal angles at two points P_0, P_3 on the same side of it, the four points are concyclic.



a circle (C_1) passes through the points P_0, P_1, P_2, P_3 since $P_0P_1P_2P_3$ is cyclic (ii)

If a circle contains three points, then only one circle may satisfy those three points



only one circle C_1 can satisfy all three points P_0, P_1, P_2

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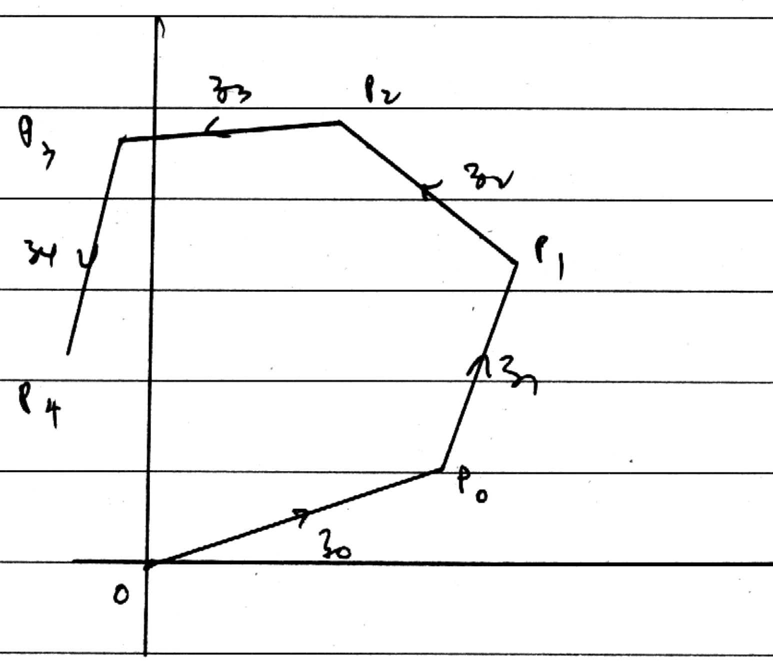
consider a circle C_2 that passes through $O P_0 P_1 P_2$, since $O P_0 P_1 P_2$ is cyclic by (i).

only one circle can satisfy the three points P_0 P_1 P_2. But by previous argument, this circle must be C_1. ∴ because $O P_0 P_1 P_2$ is cyclic, and $P_0 P_1 P_2 P_3$ is cyclic, and both circles must share the same circle C_1, then O, P_0, P_1, P_2, P_3 are concyclic

ie $O P_0 P_1 P_2 P_3$ is a circle.

(iv) (i) $z_0 + z_1 + z_2 + z_3 + z_4 = 0$



$|z_0 + z_1 + z_2 + z_3 + z_4| = 0$

$\therefore$ ~~OP4~~ $\overrightarrow{OP_4} = 0$

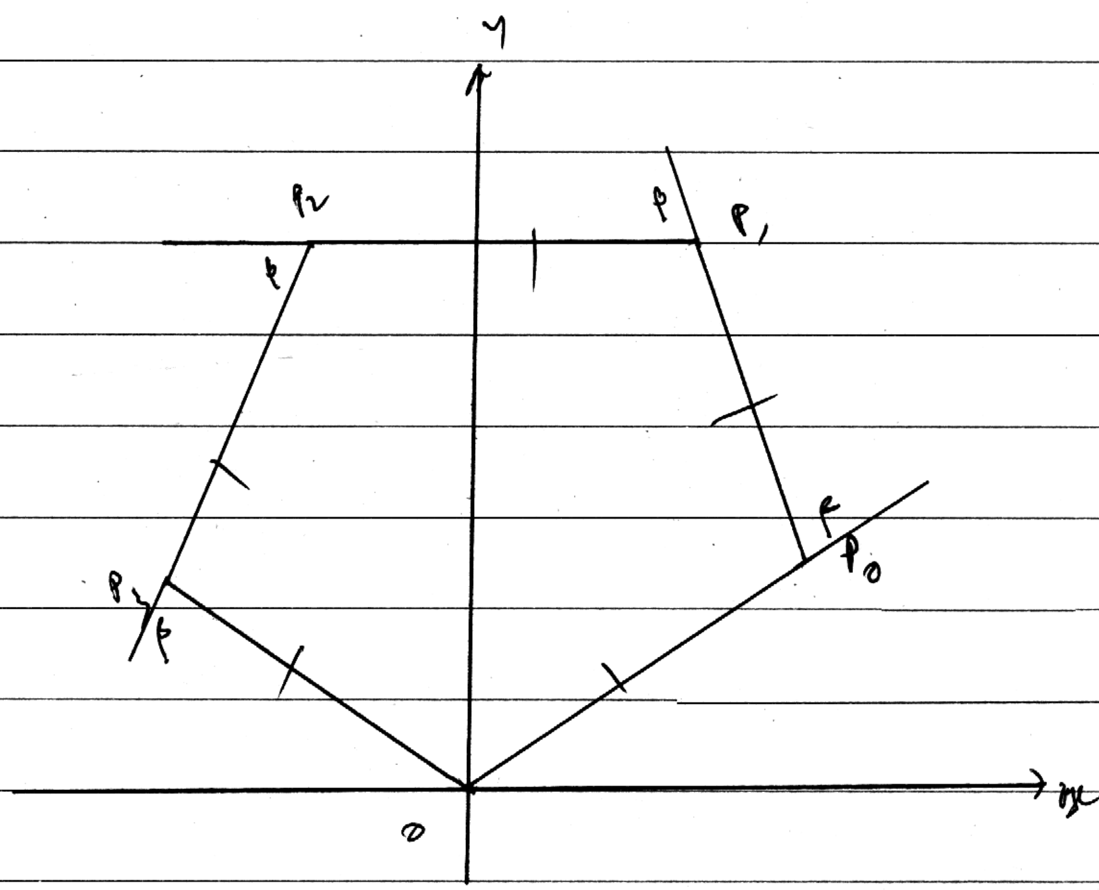
ie O and P_4 are the same point.

$\therefore$ O, P_0, P_1, P_2, P_3 lies on same circle with ~~$|OP_0| = |P_0 P_1| = \ldots = |P_3 P_4|$~~

$|OP_0| = |P_0 P_1| = |P_1 P_2| = |P_2 P_3| = |P_3 P_4|$

but $|P_3 P_4| = |P_3 O|$

$\therefore$ O, P_0, P_1, P_2, P_3 represent points on a regular pentagon.



angle sum of pentagon is ~~$3\times180°$~~ ~~$=540°$~~ $(n-2)\,180°$

$= 3\times180°$

$= 540°$

since the pentagon is regular, $540°$ is distributed evenly over 5 points

$\therefore \angle P_3 O P_0 = \angle O P_0 P_1 = \angle P_0 P_1 P_2$

$= \angle P_1 P_2 P_3 = \angle P_2 P_3 P_0$

$= \frac{3\pi}{5}$

But β + (angle at vertex) $= \pi$ (adjacent angles on straight line are supplementary)

$\therefore$ ~~angle at~~

$\beta = \pi$ − angle at vertex

$= \pi - \frac{3\pi}{5}$

$= \frac{2\pi}{5}$

$\therefore \beta = \frac{2\pi}{5}$ (QED)