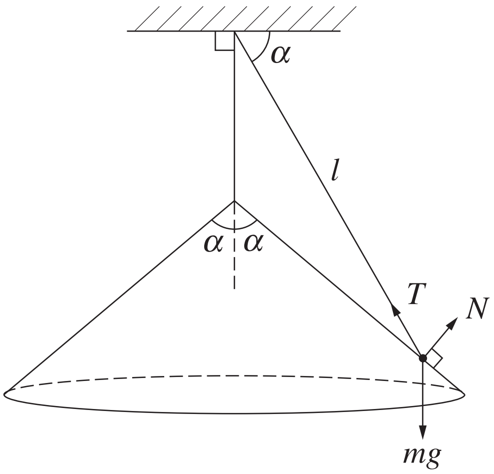
Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m is suspended by a string of length l from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity ω. The angle of the cone at its vertex is 2α, where $\alpha > \frac{\pi}{4}$, and the string makes an angle of α with the horizontal as shown in the diagram. The forces acting on the particle are the tension in the string T, the normal reaction to the cone N and the gravitational force mg.



NOT TO SCALE

(i) Show, with the aid of a diagram, that the vertical component of N is $N \sin \alpha$. **1**

(ii) Show that $T + N = \dfrac{mg}{\sin \alpha}$, and find an expression for $T - N$ in terms of m, l and ω. **3**

(iii) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone. Find an expression for this value of ω in terms of α, l and g. **2**

Question 6 continues on page 10

Marks

Question 6 (continued)

(b) For $n = 0, 1, 2, \ldots$ let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta .$$

(i) Show that $I_1 = \frac{1}{2} \ln 2$. **1**

(ii) Show that, for $n \geq 2$, **3**

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

(iii) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that **3**

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}.$$

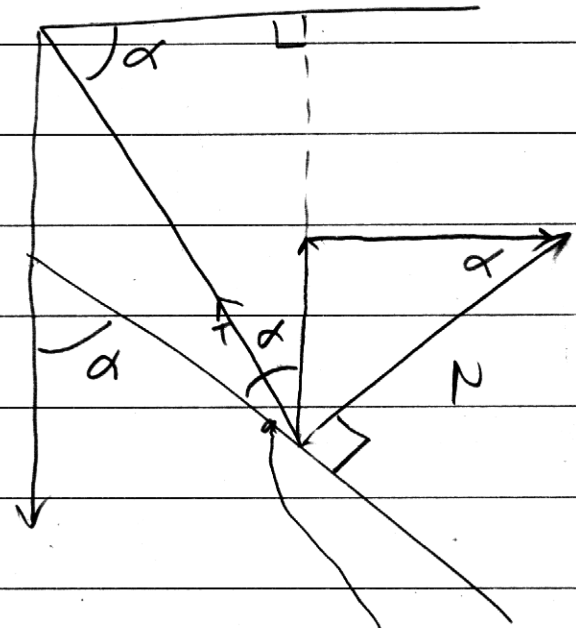
(iv) By using the recurrence relation of part (ii), find I_5 and deduce that **2**

$$\frac{2}{3} < \ln 2 < \frac{3}{4} .$$

End of Question 6

BOARD OF STUDIES
NEW SOUTH WALES

i)



alternate ∠'s so is α

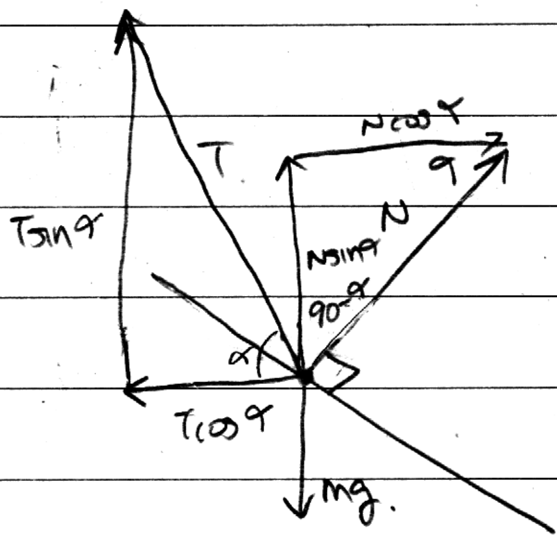
∵ Perpendicular then adj ∠ is 90 − α

∴ Labelled ∠ = α (∠ sum of Δ)

Hence $\sin\alpha = \dfrac{\text{Vertical component of } N}{N}$.

Hence vertical component = $N\sin\alpha$.

ii)



Resolving forces vertically,

$T\sin\alpha + N\sin\alpha = mg.$

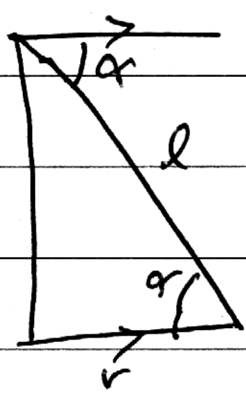
$\therefore \quad T + N = \dfrac{mg}{\sin\alpha}$

Resolving forces horizontally,

$T\cos\alpha - N\cos\alpha = m\omega^2 r$. (centripedal force)

$\therefore \; T\cos\alpha - N\cos\alpha = m\omega^2 l\cos\alpha$

For $\cos\alpha \neq 0$, $\quad T - N = m\omega^2 l$.



By trig, $r = l\cos\alpha$.

BOARD OF STUDIES
NEW SOUTH WALES

(ii) From $T+N = \dfrac{mg}{\sin\theta}$

& $T-N = m\omega^2 l.$

$$2T = \frac{mg}{\sin\theta} + m\omega^2 l.$$

$$T = \frac{1}{2}\left(\frac{mg}{\sin\theta} + m\omega^2 l\right)$$

Sub this into $T-N = m\omega^2 l.$

$$\frac{1}{2}\left(\frac{mg}{\sin\theta} + m\omega^2 l\right) - N = m\omega^2 l.$$

Sub $N = 0.$

$$\therefore \frac{mg}{2\sin\theta} + \frac{1}{2}m\omega^2 l = m\omega^2 l.$$

$$\frac{1}{2}m\omega^2 l = \frac{mg}{2\sin\theta}.$$

For $m \neq 0.$ $\omega^2 l = \dfrac{g}{\sin\theta}.$

$$\omega^2 = \frac{g}{l\sin\theta}$$

$$\omega = \sqrt{\frac{g}{l\sin\theta}}.$$

b). $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$.

$$I_1 = \int_0^{\pi/4} \tan\theta \, d\theta$$

$$= -\int_0^{\pi/4} \frac{-\sin\theta}{\cos\theta} \, d\theta$$

$$= \Big[-\ln|\cos\theta| \Big]_0^{\pi/4}.$$

$$= -\ln(\cos \tfrac{\pi}{4}) + \ln(\cos 0|$$

$$= -\ln \frac{1}{\sqrt{2}} + \ln 1$$

$$= \ln\sqrt{2} + 0.$$

$$= \ln 2^{1/2}$$

$$= \tfrac{1}{2}\ln 2.$$

ii) $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta \tan^2\theta \, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta \, (\sec^2\theta - 1) \, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta \, d\theta - \int_0^{\pi/2} \tan^{n-2}\theta \, d\theta$.

$= \frac{1}{n-1} \left[\tan^{n-1}\theta\right]_0^{\pi/4} - I_{n-2}$.

ie. $I_n + I_{n-2} = \frac{1}{n-1} \left[\tan^{n-1}(\pi/4) - \tan^{n-1}0\right]$

$= \frac{1}{n-1} (1-0)$

$= \frac{1}{n-1}$.

$\therefore I_n + I_{n-2} = \frac{1}{n-1}$.

iii) ~~the~~ Let us consider the area under the curve. For $0<x<\frac{\pi}{4}$, $\tan x<1$.

$\therefore$ For $0<x<\frac{\pi}{4}$, $\tan^n x < \tan^{n-1} x$

$< \tan^{n-2} x$.

$\therefore \tan^n x < \tan^{n-2} x$.

The $\int_0^{\pi/4} \tan^n \theta \, d\theta$ represents the area under the curve $\tan^n \theta$ between 0 & $\frac{\pi}{4}$, & as the power increases, the smaller the area. $\therefore \int_0^{\pi/4} \tan^n \theta \, d\theta \leq \int_0^{\pi/4} \tan^{n-2} \theta \, d\theta$

$$I_n \leq \int_0^{\pi/4} \tan^{n-2} \theta \, d\theta.$$

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

$$I_{n-2} = \frac{1}{n-1} - I_n.$$

$$\because I_n \leq I_{n-2}$$
$$= \frac{1}{n-1} - I_n.$$

$$\therefore 2I_n \leq \frac{1}{n-1}$$

$$\& \quad I_n \leq \frac{1}{2(n-1)}$$

Consider the left hand side of the inequality,

$$I_n \geq I_{n+2}.$$

$$I_{n+2} + I_n = \frac{1}{(n+2)-1}$$

$$= \frac{1}{n+1}$$

$$\therefore \quad I_n \geq I_{n+2}$$

& $I_{n+2} = \dfrac{1}{n+1} - I_n$

Then $I_n \geq \dfrac{1}{n+1} - I_n$

ie. $2I_n \geq \dfrac{1}{n+1}$

$$I_n \geq \frac{1}{2(n+1)}.$$

$$\therefore \frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

BOARD OF STUDIES
NEW SOUTH WALES

iv) $I_5 + I_3 = \frac{1}{4}$.

$I_3 + I_1 = \frac{1}{2}$.

$\therefore \quad I_5 - I_1 = \frac{1}{4} - \frac{1}{2}$

$= -\frac{1}{4}$.

$\because \quad I_1 = \frac{1}{2} \ln 2$.

Then $I_5 = -\frac{1}{4} + \frac{1}{2} \ln 2$

$= \frac{1}{2} (\ln 2 - \frac{1}{2})$.

We know $\frac{1}{2(5+1)} < I_5 < \frac{1}{2(5-1)}$

$\frac{1}{12} < \frac{1}{2}(\ln 2 - \frac{1}{2}) < \frac{1}{8}$.

$\frac{1}{6} < \ln 2 - \frac{1}{2} < \frac{1}{4}$.

$\cancel{\frac{7}{6} < \ln 2 < \frac{5}{4}}$ $\quad -3$.

$\cancel{\therefore \frac{7}{6} < \ln 2 < \frac{5}{4}}$

$\therefore \quad \frac{2}{3} < \ln 2 < \frac{3}{4}$