**Marks**

**Question 5** (15 marks) Use a SEPARATE writing booklet.

(a) The equation $4x^3 - 27x + k = 0$ has a double root. Find the possible values of $k$. **2**

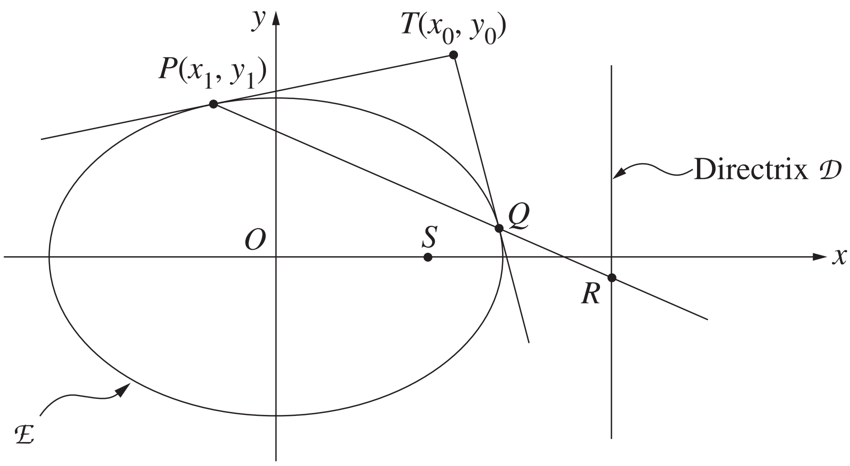
(b) Let $\alpha$, $\beta$, and $\gamma$ be the roots of the equation $x^3 - 5x^2 + 5 = 0$.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha - 1$, $\beta - 1$, and $\gamma - 1$. **2**

(ii) Find a polynomial equation with integer coefficients whose roots are $\alpha^2$, $\beta^2$, and $\gamma^2$. **2**

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

(c)




The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and focus $S$ and directrix $\mathcal{D}$ as shown in the diagram. The point $T(x_0, y_0)$ lies outside the ellipse and is not on the $x$ axis. The chord of contact $PQ$ from $T$ intersects $\mathcal{D}$ at $R$, as shown in the diagram.

(i) Show that the equation of the tangent to the ellipse at the point $P(x_1, y_1)$ is **2**

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

(ii) Show that the equation of the chord of contact from $T$ is **2**

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

(iii) Show that $TS$ is perpendicular to $SR$. **3**

(a) Let $P(x) = 4x^3 - 27x + k$

$P'(x) = 12x^2 - 27$

$12x^2 - 27 = 0$

$x^2 = \frac{27}{12}$

$x = \pm\sqrt{\frac{27}{12}}$

$x = \sqrt{\frac{27}{12}}$ $\quad 4 \cdot \frac{27}{12}\sqrt{\frac{27}{12}} - 27\sqrt{\frac{27}{12}} + k = 0$

$k = 18\sqrt{\frac{27}{12}}$

$x = -\sqrt{\frac{27}{12}}$ $\quad 4 \cdot -\frac{27}{12}\sqrt{\frac{27}{12}} + 27\sqrt{\frac{27}{12}} + k = 0$

$k = -18\sqrt{\frac{27}{12}}$

$\therefore k = \pm 18\sqrt{\frac{27}{12}}$

(b) Let $P(x) = x^3 - 5x^2 + 5$

(i) $P(x+1) = (x+1)^3 - 5(x+1)^2 + 5$

$(x+1)^3 - 5(x+1)^2 + 5 = 0$

$x^3 + 3x^2 + 3x + 1 - 5x^2 - 10x - 5 + 5 = 0$

$x^3 - 2x^2 - 7x + 1 = 0$

(ii) $P(\sqrt{x}) = (\sqrt{x})^3 - 5(\sqrt{x})^2 + 5$

$x\sqrt{x} - 5x + 5 = 0$

$x\sqrt{x} = 5x - 5$

$x^3 = 25x^2 - 50x + 25$

$x^3 - 25x^2 + 50x - 25 = 0$

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(iii) $\alpha+\beta+\gamma = 5$

$\alpha\beta+\beta\gamma+\gamma\alpha = 0$

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$= 5^2 - 0$$
$$= 25$$

$$\alpha^3 = 5\alpha^2 - 5$$
$$\beta^3 = 5\beta^2 - 5$$
$$\gamma^3 = 5\gamma^2 - 5$$
$$\alpha^3+\beta^3+\gamma^3 = 5(\alpha^2+\beta^2+\gamma^2) - 15$$
$$= 5\times 25 - 15$$
$$= 110$$

(c)(i) $\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{2x}{a^2}\times\frac{b^2}{2y}$$
$$= -\frac{b^2x}{a^2y}$$
$$= -\frac{b^2x_1}{a^2y_1} \text{ at } P$$

Tangent at P:

$$y-y_1 = -\frac{b^2x_1}{a^2y_1}(x-x_1)$$
$$\frac{y_1y}{b^2} - \frac{y_1^2}{b^2} = -\frac{x_1x}{a^2} + \frac{x_1^2}{a^2}$$
$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2}$$
$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \quad \text{since } P \text{ lies on } E$$

(ii) Since $T(x_0, y_0)$ lies on this

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1$$

Which means $P(x_1, y_1)$ lies on

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Similarly tangent at $Q$ $(x_2, y_2)$ is

$$\frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 1$$

Since $T(x_0, y_0)$ lies on this

$$\frac{x_2 x_0}{a^2} + \frac{y_2 y_0}{b^2} = 1$$

And $Q$ lies on

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + 1$$

Since only one line pass through both $P$ & $Q$, this must be chord $PQ$.

(iii) Let $S$ be $(ae, 0)$ & $R$ be $\left(\frac{a}{e}, y_4\right)$

~~Tangent at~~

Chord $PQ$ meets $D$ at

$$\frac{x_0}{a^2} \cdot \frac{a}{e} + \frac{y_0 y}{b^2} = 1$$

$$\frac{x_0}{ae} + \frac{y_0 y}{b^2} = 1$$

$$\therefore \quad y_4 = \frac{b^2}{y_0}\left(1 - \frac{x_0}{ae}\right)$$

Gradient of $TS$

$$M_{TS} = \frac{y_0 - 0}{x_0 - ae}$$

Gradient of SR:

$$M_{SR} = \frac{\frac{b^2}{y_0}\left(1-\frac{x_0}{ae}\right) - 0}{\frac{a}{e} - ae}$$

$$= \frac{\frac{b^2}{y_0}\left(\frac{ae-x_0}{ae}\right)}{\frac{a-ae^2}{e}}$$

$$= \frac{b^2(ae-x_0)}{y_0 a^2(1-e^2)}$$

$$= \frac{ae-x_0}{y_0} \quad \text{since } b^2 = a^2(1-e^2)$$

$$\therefore M_{TS} \times M_{SR} = \frac{y_0}{x_0-ae} \times \frac{ae-x_0}{y_0}$$

$$= -1$$

$\therefore TS \perp SR$