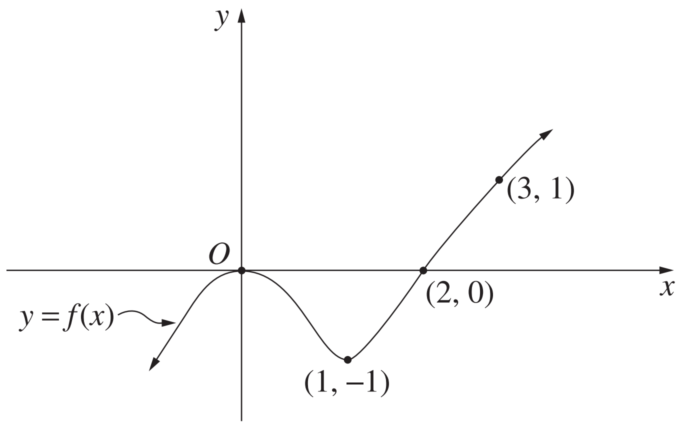
Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graph of $y = f(x)$.

Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \left| f(|x|) \right|$ **2**

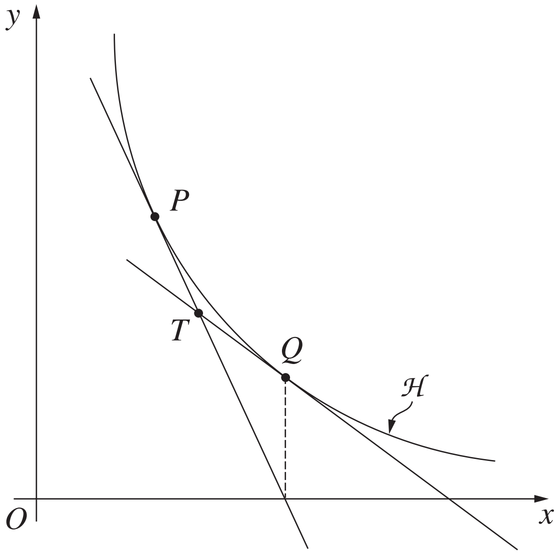
(iv) $y = \ln\left(f(x)\right)$. **2**

Question 3 continues on page 5

Marks

Question 3 (continued)

(b)



The distinct points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are on the same branch of the hyperbola $\mathcal{H}$ with equation $xy = c^2$. The tangents to $\mathcal{H}$ at P and Q meet at the point T.

(i) Show that the equation of the tangent at P is **2**

$$x + p^2 y = 2cp.$$

(ii) Show that T is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

(iii) Suppose P and Q move so that the tangent at P intersects the x axis at $(cq, 0)$. **3**

Show that the locus of T is a hyperbola, and state its eccentricity.

End of Question 3

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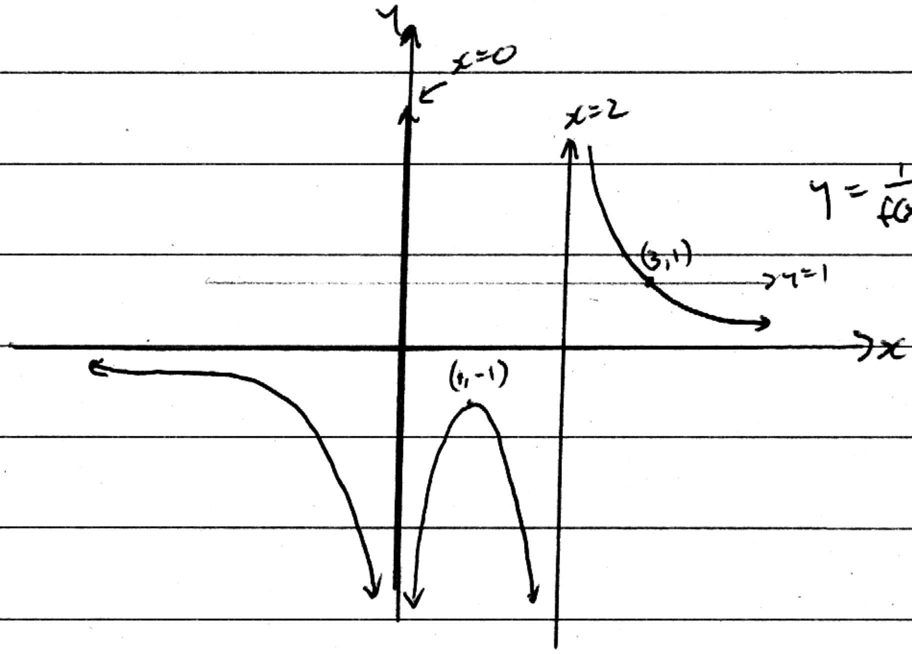
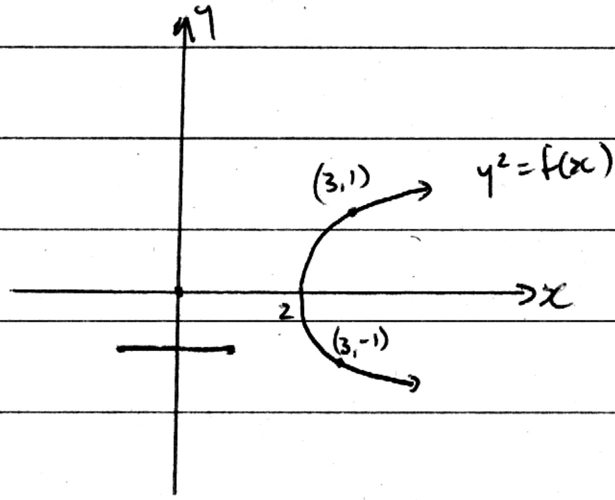
Question 3.

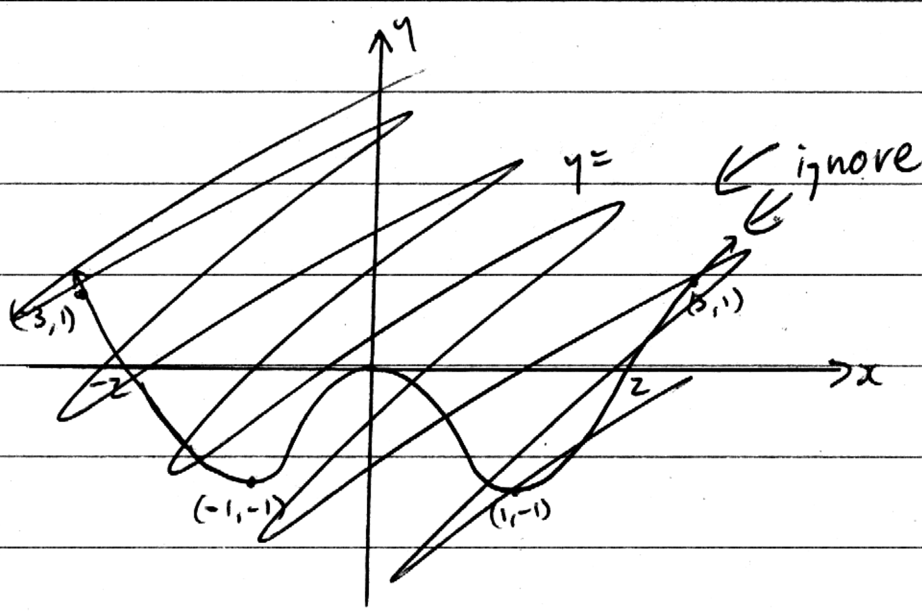
a) i



ii



iii



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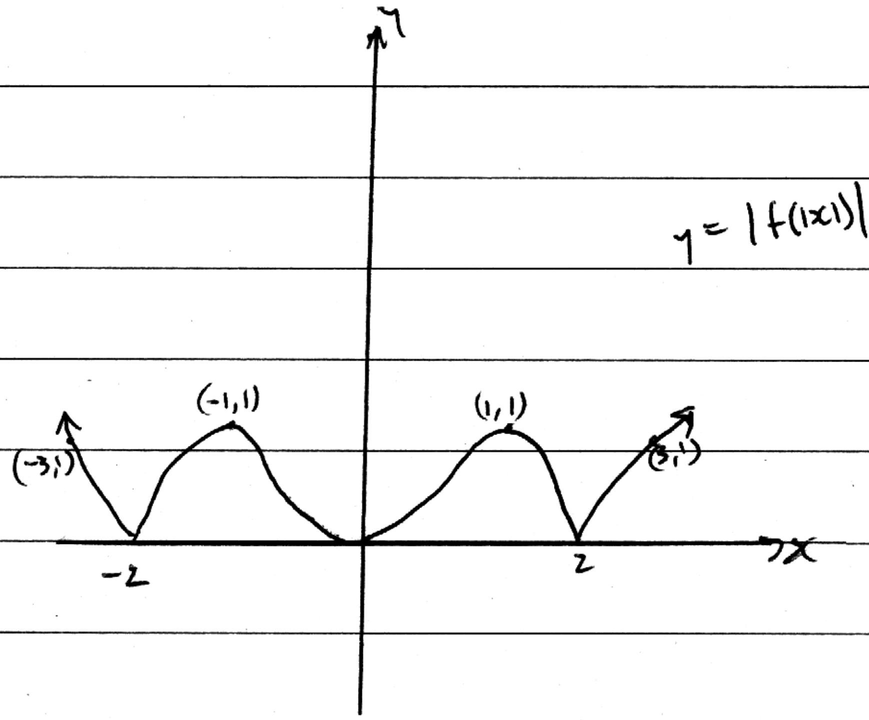
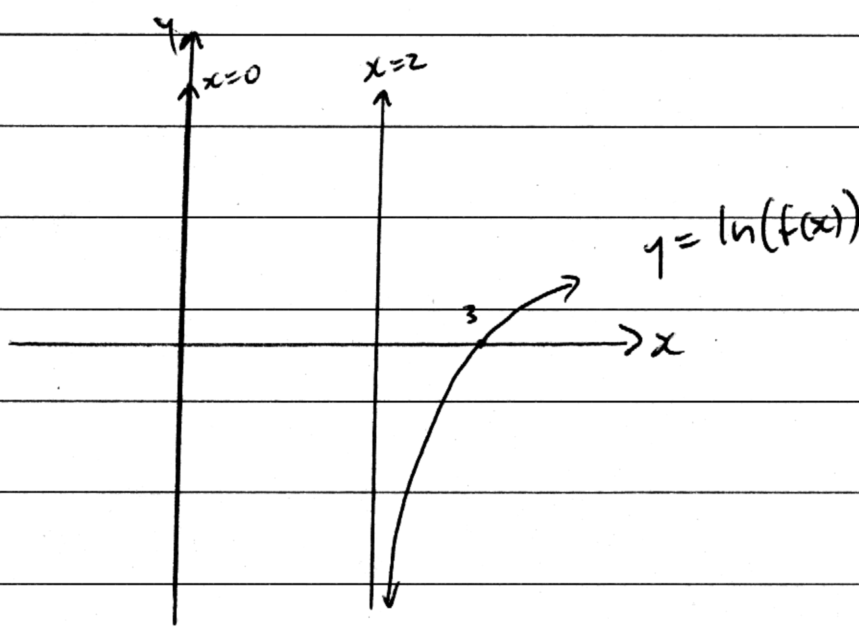
a) iii



iv



b) i ~~on~~ ~~$\frac{d}{dx}$~~ differentiating implicitly,

$$y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow @ P, \ \frac{dy}{dx} = -\frac{1}{p^2}$$

Now $y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$$p^2y - cp = -x + cp$$

$$\therefore x + p^2y = 2cp. \quad -①$$

ii Similarly, $x + q^2y = 2cq$

Solving simultaneously,

$$p^2y - 2cp = q^2y - 2cq$$

$$y(p^2 - q^2) = 2c(p - q)$$

$$\therefore y = \frac{2c}{p+q}$$

into ①, $x + \frac{2cp^2}{p+q} = 2cp$

$$x = \frac{2cp(p+q) - 2cp^2}{p+q}$$

$$= \frac{2cpq}{p+q}$$

$$\therefore T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

iii ie @ $y=0$, $x = cq$ but from ①, $x = 2cp$ @ $y = 0$

$$\therefore cq = 2cp \Rightarrow q = 2p$$

P.T.O

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NEW SOUTH WALES

So $q = 2p$, Now $T\left(\frac{4cp^2}{3p}, \frac{2c}{3p}\right)$

$$x = \frac{4cp}{3}$$

$$\therefore \; p = \frac{3x}{4c} \quad -③ \qquad \text{Now} \quad y = \frac{2}{3} \cdot \frac{c}{p}$$

$$= \frac{2}{3} \cdot \frac{4c^2}{3x} \quad \text{from ③}$$

$$\therefore \; xy = \frac{8c^2}{9}$$

This is the locus of a hyperbola.

The eccentricity of the hyperbola is $e = \sqrt{2}$.