Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 1 + 2i$ and $w = 1 + i$. Find, in the form $x + iy$,

(i) $z\overline{w}$ **1**

(ii) $\frac{1}{w}$. **1**

(b) On an Argand diagram, shade in the region where the inequalities **3**

$$0 \le \operatorname{Re} z \le 2 \quad \text{and} \quad |z - 1 + i| \le 2$$

both hold.

(c) It is given that $2 + i$ is a root of

$$P(z) = z^3 + rz^2 + sz + 20,$$

where r and s are real numbers.

(i) State why $2 - i$ is also a root of $P(z)$. **1**

(ii) Factorise $P(z)$ over the real numbers. **2**

(d) Prove by induction that, for all integers $n \ge 1$, **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta).$$

(e) Let $z = 2(\cos\theta + i\sin\theta)$.

(i) Find $\overline{1 - z}$. **1**

(ii) Show that the real part of $\frac{1}{1-z}$ is $\frac{1 - 2\cos\theta}{5 - 4\cos\theta}$. **2**

(iii) Express the imaginary part of $\frac{1}{1-z}$ in terms of θ. **1**

Q2.

a) $z = 1+2i \qquad \omega = 1+i$

i) $z\overline{\omega} = (1+2i)(1-i)$.

$= 1 - 2\overset{-1}{i^2} - i + 2i$.

$= 3 + i$.

ii) $\dfrac{1}{\omega} = \dfrac{1}{1+i} \cdot \dfrac{1-i}{1-i}$

$1^2 - i^2$

$1 - i^2$

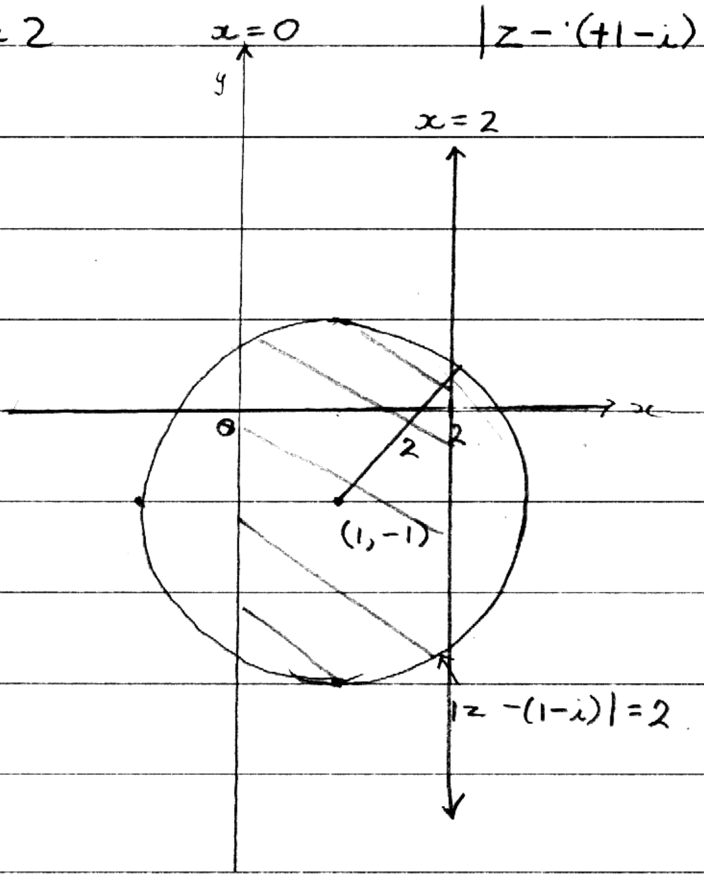
$1 - -1$.

$= \dfrac{1-i}{2}$.

$= \dfrac{1}{2} - \dfrac{i}{2}$.

b) $0 \leq \text{Re}\, z \leq 2 \qquad x=0 \qquad |z-(+1-i)| \leq 2$

$0 \leq x \leq 2$



c) $P(z) = z^3 + rz^2 + sz + 20$

i) if a ~~complex~~ polynomial with real coefficients has a complex root, then its conjugate is also a root. Since $P(z)$ has real coeff. r, s, ~~the~~ if $2+i$ is a root, $2-i$ is also a root.

ii) $\therefore \quad (z-(2+i))(z-(2-i))$ is a factor. $\quad 4^2 - i^2$

ie. $z^2 - 4z + 5$ is a factor

$P(z) = (z^2 - 4z + 5)(z+4)$.

d) test $n=1$ $\quad LHS = (\cos\theta - i\sin\theta)^1$

$RHS = \cos(1\cdot\theta) - i\sin(1\cdot\theta) = LHS$.

$\therefore$ true for $n=1$.

assume true for $n=k$.

ie. $(\cos\theta - i\sin\theta)^k = \cos(k\theta) - i\sin(k\theta)$.

prove true for $n=k+1$.

ie. $(\cos\theta - i\sin\theta)^{k+1} = \cos((k+1)\theta) - i\sin((k+1)\theta)$.

$LHS = (\cos\theta - i\sin\theta)^k(\cos\theta - i\sin\theta)$

$= (\cos(k\theta) - i\sin(k\theta))(\cos\theta - i\sin\theta)$ [using assumption]

$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta \mp i\,(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta)$.

$= \cos(k\theta + \theta) - i\sin(k\theta + \theta)$.

$= \cos(\theta(k+1)) - i\sin((k+1)\theta) = RHS$

$\therefore$ true for $n = k+1$.

$\therefore$ if true for $n=k$, it is true for $n=k+1$

Since true for $n=1$, it is true for $n=2,3,4\ldots$

ie. all integers $n \geq 1$

e) $z = 2(\cos\theta + i\sin\theta)$.

i) $\overline{1-z} = \overline{1 - 2\cos\theta - 2i\sin\theta}$.

$= 1 - 2\cos\theta + 2i\sin\theta$.

ii) $\frac{1}{1-z} = \frac{\overline{1-z}}{(1-z)(\overline{1-z})}$

$= \frac{1-2\cos\theta+2i\sin\theta}{(1-2\cos\theta)^2-4\overset{-1}{i^2}\sin^2\theta}$

$= \frac{1-2\cos\theta+2i\sin\theta}{(4\cos^2\theta)-4\cos\theta+1+(4\sin^2\theta)}$

$= \frac{1-2\cos\theta+2i\sin\theta}{\cancel{4\cos^2}\ 5-4\cos\theta}$

$\therefore$ Real part of $\frac{1}{1-z} = \frac{1-2\cos\theta}{5-4\cos\theta}$.

iii) $\text{Im}\left(\frac{1}{1-z}\right) = \frac{2\sin\theta}{5-4\cos\theta}$