

**Total marks – 120**  
**Attempt Questions 1–8**  
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

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**Question 1** (15 marks) Use a SEPARATE writing booklet. **Marks**

- (a) By using the substitution  $u = \sec x$ , or otherwise, find **2**

$$\int \sec^3 x \tan x \, dx .$$

- (b) By completing the square, find  $\int \frac{dx}{x^2 + 2x + 2}$ . **2**

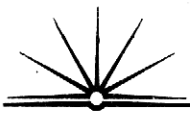
- (c) Find  $\int \frac{x \, dx}{(x+3)(x-1)}$ . **3**

- (d) By using two applications of integration by parts, evaluate **4**

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx .$$

- (e) Use the substitution  $t = \tan \frac{\theta}{2}$  to find **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \cos \theta} .$$



Q 1

$$a) \int \sec^3 x \tan x \, dx = \int \frac{u^2 (\sec x \tan x) \, du}{\sec x \tan x}$$

$$\text{let } u = \sec x \qquad = \frac{u^3}{3} + c$$

$$\frac{du}{dx} = \sec x \tan x.$$

$$= \frac{\sec^3 x}{3} + c$$

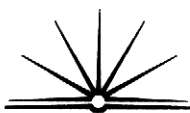
or

$$b) \int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(x+1)^2 + 1}$$

$$= \tan^{-1}(x+1) + c$$

$$(x+1)^2$$

$$= x^2 + 2x + 1$$



$$c) \int \frac{x dx}{(x+3)(x-1)} = \int \frac{A}{x+3} + \frac{B}{x-1} dx.$$

$$x = A(x-1) + B(x+3).$$

$$\text{let } x = 1$$

$$1 = 0 + 4B.$$

$$B = \frac{1}{4}$$

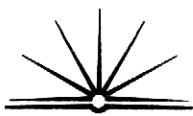
$$x = A(x-1) + \frac{1}{4}(x+3).$$

$$\text{let } x = -3.$$

$$-3 = -4A + 0$$

$$A = \frac{3}{4}$$

$$\begin{aligned} \therefore \int \frac{x dx}{(x+3)(x-1)} &= \int \frac{3}{4(x+3)} dx + \int \frac{1}{4(x-1)} dx \\ &= \frac{3}{4} \ln|x+3| + \frac{1}{4} \ln|x-1| + C \end{aligned}$$



d)

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$\text{let } u = \cos x$$

$$\underline{du} = -\sin x \, dx.$$

$$\int u \, dv = uv - \int v \, du$$

$$\text{let } dv = e^x$$

$$= [e^x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx \quad v = e^x$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$\text{let } u = \sin x$$

$$du = \cos x \, dx.$$

$$= [e^x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx.$$

$$\text{let } dv = e^x$$

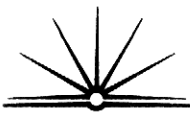
$$v = e^x$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \cos x \, dx = [e^x \cos x]_0^{\frac{\pi}{2}} + [e^x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$2 \int_0^{\frac{\pi}{2}} e^x \cos x \, dx = 0 - [1] + [e^{\frac{\pi}{2}}] - [0].$$

$$2 \int_0^{\frac{\pi}{2}} e^x \cos x \, dx = e^{\frac{\pi}{2}} - 1$$

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \frac{e^{\frac{\pi}{2}} - 1}{2}$$



$$e) \int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \cos\theta} = \int_0^1 \frac{1}{2 + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$\text{let } t = \tan \frac{\theta}{2} = \int_0^1 \frac{1}{2 + 2t^2 + 1 - t^2} \times 2dt$$

$$\frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{\theta}{2} = \int_0^1 \frac{1}{3 + t^2} \times 2dt$$

$$d\theta = \frac{2dt}{1+t^2} = \int_0^1 \frac{2dt}{3+t^2}$$

$$\text{when } \theta = \frac{\pi}{2}$$

$$t = 1$$

$$\text{when } \theta = 0$$

$$t = 0$$

$$= \left[ 2 \times \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \right]_0^1$$

$$= \left[ \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} \right] - \left[ \frac{2}{\sqrt{3}} \tan^{-1} 0 \right]$$

$$= \left[ \frac{2}{\sqrt{3}} \times \frac{\pi}{6} \right] - [0]$$

$$= \frac{2\pi}{6\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}}$$