Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Let m be a positive integer.

(i) By using De Moivre's theorem, show that **2**

$$\sin(2m+1)\theta = \binom{2m+1}{1}\cos^{2m}\theta\sin\theta - \binom{2m+1}{3}\cos^{2m-2}\theta\sin^3\theta + \ldots + (-1)^m\sin^{2m+1}\theta.$$

(ii) Deduce that the polynomial **3**

$$p(x) \equiv \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \ldots + (-1)^m$$

has m distinct roots

$$\alpha_k = \cot^2\left(\frac{k\pi}{2m+1}\right) \quad \text{where } k = 1, 2, \ldots, m.$$

(iii) Prove that **2**

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \ldots + \cot^2\left(\frac{m\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

(iv) You are given that $\cot\theta < \dfrac{1}{\theta}$ for $0 < \theta < \dfrac{\pi}{2}$. **2**

Deduce that

$$\frac{\pi^2}{6} < \left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{m^2}\right)\frac{(2m+1)^2}{2m(2m-1)}.$$

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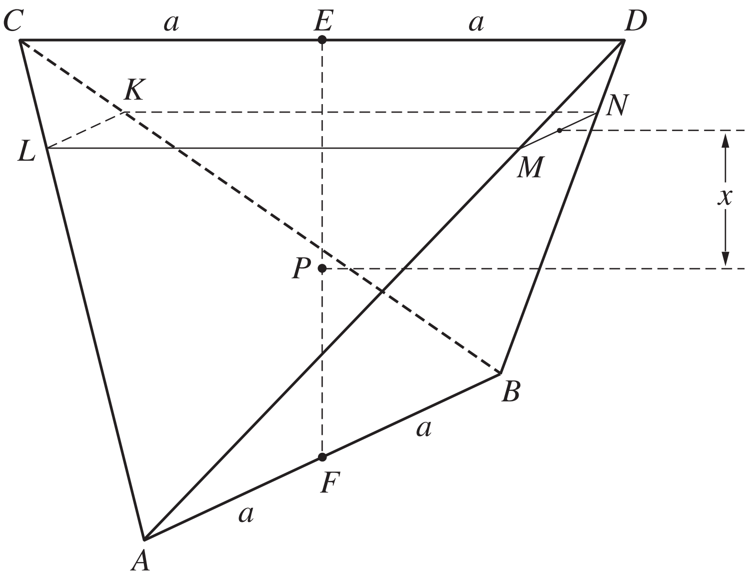
Marks

Question 8 (continued)

(b)



In the diagram, AB and CD are line segments of length $2a$ in horizontal planes at a distance $2a$ apart. The midpoint E of CD is vertically above the midpoint F of AB, and AB lies in the South–North direction, while CD lies in the West–East direction.

The rectangle $KLMN$ is the horizontal cross-section of the tetrahedron $ABCD$ at distance x from the midpoint P of EF (so $PE = PF = a$).

(i) By considering the triangle ABE, deduce that $KL = a - x$, and find the area of the rectangle $KLMN$. **4**

(ii) Find the volume of the tetrahedron $ABCD$. **2**

End of paper