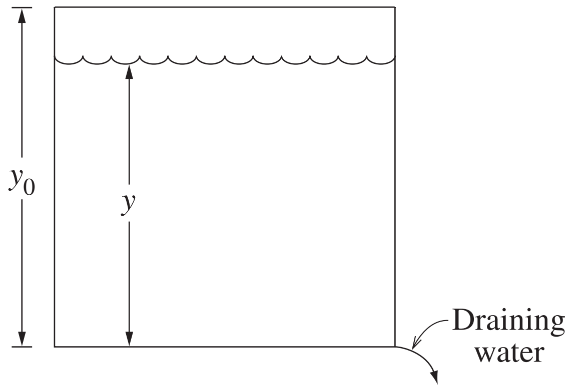
Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram represents a vertical cylindrical water cooler of constant cross-sectional area A. Water drains through a hole at the bottom of the cooler. From physical principles, it is known that the volume V of water decreases at a rate given by

$$\frac{dV}{dt} = -k\sqrt{y},$$

where k is a positive constant and y is the depth of water.

Initially the cooler is full and it takes T seconds to drain. Thus $y = y_0$ when $t = 0$, and $y = 0$ when $t = T$.

(i) Show that $\dfrac{dy}{dt} = -\dfrac{k}{A}\sqrt{y}$. **1**

(ii) By considering the equation for $\dfrac{dt}{dy}$, or otherwise, show that **4**

$$y = y_0\left(1 - \frac{t}{T}\right)^2 \quad \text{for } 0 \le t \le T.$$

(iii) Suppose it takes 10 seconds for half the water to drain out. How long does it take to empty the full cooler? **2**

Question 7 continues on page 12

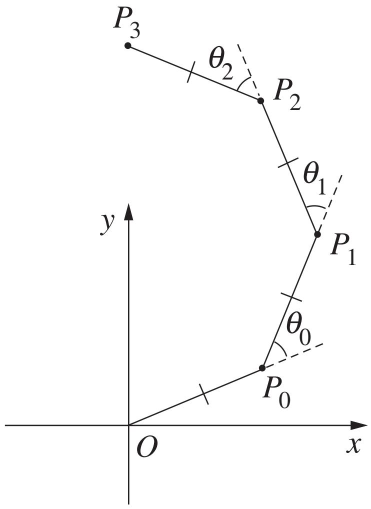
Marks

Question 7 (continued)

(b) Suppose $0 < \alpha,\ \beta < \dfrac{\pi}{2}$ and define complex numbers z_n by

$$z_n = \cos(\alpha + n\beta) + i\sin(\alpha + n\beta)$$

for $n = 0, 1, 2, 3, 4$. The points P_0, P_1, P_2 and P_3 are the points in the Argand diagram that correspond to the complex numbers z_0, $z_0 + z_1$, $z_0 + z_1 + z_2$ and $z_0 + z_1 + z_2 + z_3$ respectively. The angles θ_0, θ_1 and θ_2 are the external angles at P_0, P_1 and P_2 as shown in the diagram below.



(i) Using vector addition, explain why **2**

$$\theta_0 = \theta_1 = \theta_2 = \beta.$$

(ii) Show that $\angle P_0OP_1 = \angle P_0P_2P_1$, and explain why $OP_0P_1P_2$ is a cyclic quadrilateral. **2**

(iii) Show that $P_0P_1P_2P_3$ is a cyclic quadrilateral, and explain why the points O, P_0, P_1, P_2 and P_3 are concyclic. **2**

(iv) Suppose that $z_0 + z_1 + z_2 + z_3 + z_4 = 0$. Show that **2**

$$\beta = \frac{2\pi}{5}.$$

End of Question 7