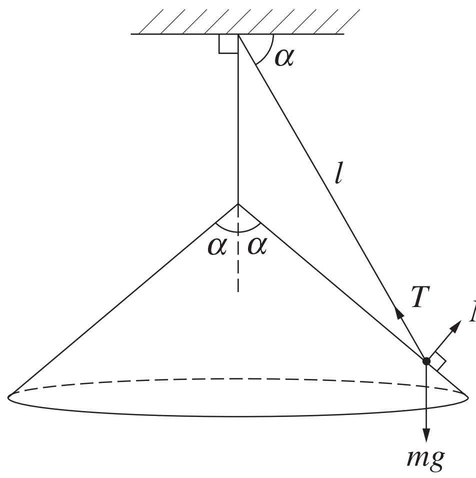
Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m is suspended by a string of length l from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity ω. The angle of the cone at its vertex is 2α, where $\alpha > \frac{\pi}{4}$, and the string makes an angle of α with the horizontal as shown in the diagram. The forces acting on the particle are the tension in the string T, the normal reaction to the cone N and the gravitational force mg.



NOT TO SCALE

(i) Show, with the aid of a diagram, that the vertical component of N is $N\sin\alpha$. **1**

(ii) Show that $T + N = \dfrac{mg}{\sin\alpha}$, and find an expression for $T - N$ in terms of m, l and ω. **3**

(iii) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone. Find an expression for this value of ω in terms of α, l and g. **2**

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Marks

Question 6 (continued)

(b) For $n = 0, 1, 2, \ldots$ let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta .$$

(i) Show that $I_1 = \frac{1}{2} \ln 2$. **1**

(ii) Show that, for $n \geq 2$, **3**

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

(iii) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that **3**

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}.$$

(iv) By using the recurrence relation of part (ii), find I_5 and deduce that **2**

$$\frac{2}{3} < \ln 2 < \frac{3}{4}.$$

End of Question 6