Question 2

(a) $z = 1 + 2i \qquad w = 1 + i \qquad \overline{w} = 1 - i$

(i) $z\overline{w} = (1+2i)(1-i)$

$= 1 + 2 + 2i - i$

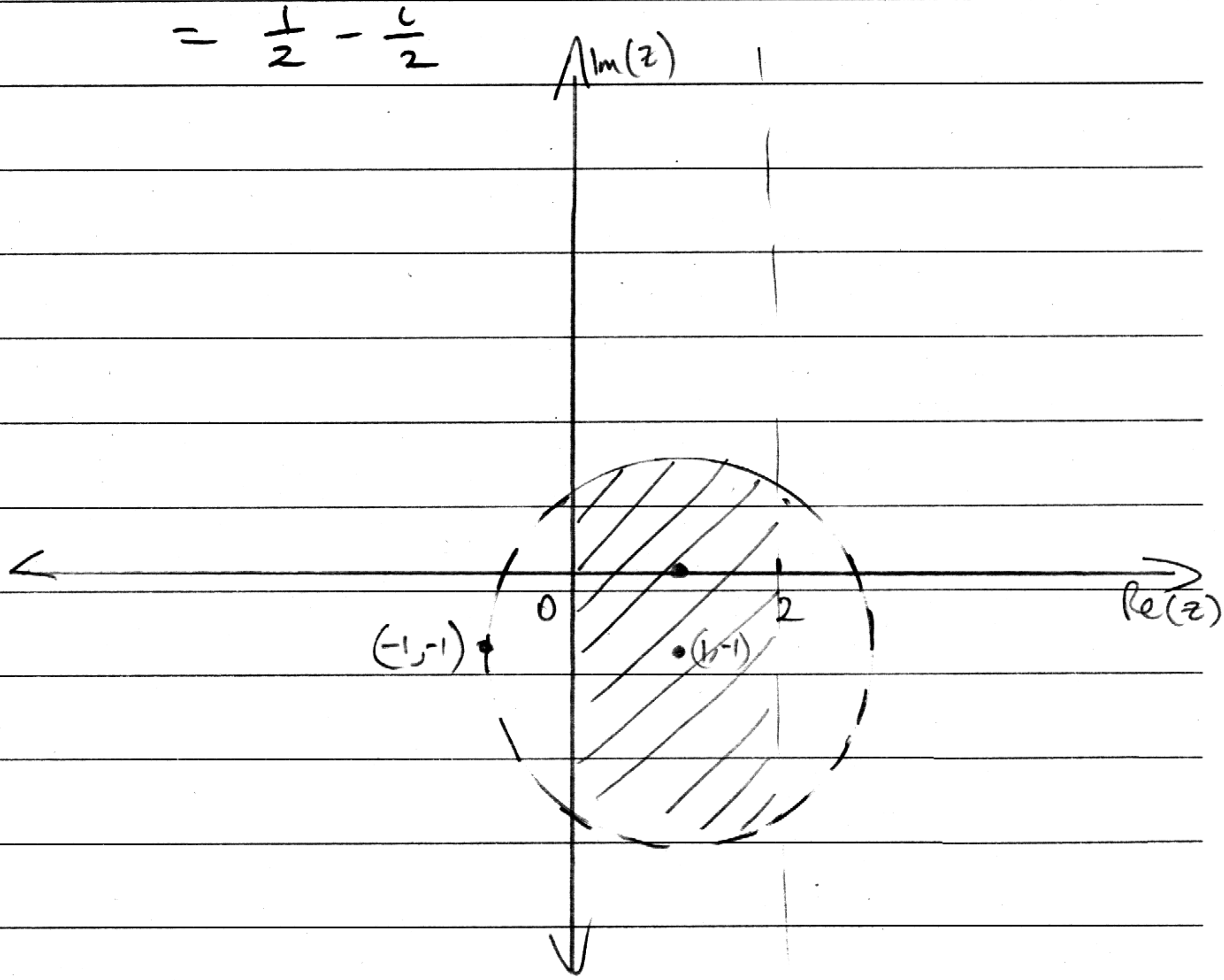
$= 3 + i$

(ii) $\dfrac{1}{w} = \dfrac{1}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{1-i}{2}$

$= \dfrac{1}{2} - \dfrac{i}{2}$

(b)



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(c) (i) $2-i$ is a root of $P(z)$ because if the coefficients are real (s and r real) ~~then~~ and $2+i$ is a root then the ~~conj~~ complex conjugate is also a root.

(ii) $P(z) = (z-2-i)(z-2+i)Q(z)$

$$P(z) = ((z-2)^2 - i^2)\, Q(z)$$
$$= (z^2 - 4z + 5)(z+4) \quad \text{by inspection}$$

(d) $(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$

For $n=1$: LHS $= \cos\theta - i\sin\theta$

RHS $= \cos\theta - i\sin\theta$

$\therefore$ LHS $=$ RHS and true for $n=1$

~~Assume its true for $n=k$ where $k \geq 1$~~

$$(\cos\theta - i\sin\theta)^k = \cos(k\theta) - i\sin(k\theta)$$

Now must prove for $n=k+1$

$$(\cos\theta - i\sin\theta)^{k+1} = \cos((k+1)\theta) - i\sin((k+1)\theta)$$

$$\cos((k+1)\theta) - i\sin((k+1)\theta)$$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta - i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

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$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta - i\sin(k\theta)\cos\theta - i\cos(k\theta)\sin\theta$

$= \cos\theta(\cos k\theta - i\sin k\theta) - \sin\theta(\sin(k\theta) + i\cos(k\theta))$

$= \cos\theta((\cos\theta - i\sin\theta)^k - \sin\theta(\sin(k\theta) + i\cos(k\theta)))$

$\cancel{= \cos\theta(\cos\theta - i\sin\theta)^k - \sin\theta(\cos(\tfrac{\pi}{2} - k\theta) + i\sin(\tfrac{\pi}{2} - k\theta)}$

$= \cos\theta(\cos\theta - i\sin\theta)^k - i\sin k\theta(\cos$

$= \cos(k\theta)(\cos\theta - i\sin\theta) + \cancel{\sin(k\theta(\sin\theta - i\cos\theta)}$

$\cancel{\sin(k\theta(-\sin\theta - i\cos\theta)}$

$\sin\theta(-\sin(k\theta) + i\cos(k\theta))$

(e) $z = 2(\cos\theta + i\sin\theta)$

(i) $1 - z = 1 - 2\cos\theta + \cancel{} \cdot 2i\sin\theta$

$\overline{1-z} = 1 - 2\cos\theta - 2i\sin\theta$

(ii) $\dfrac{1}{1-z} = \dfrac{1}{1 - 2\cos\theta + 2i\sin\theta} \times \dfrac{1 - 2\cos\theta - 2i\sin\theta}{1 - 2\cos\theta - 2i\sin\theta}$

$= \dfrac{1 - 2\cos\theta - 2i\sin\theta}{(1 - 2\cos\theta)^2 + 4\sin^2\theta}$

$\therefore \operatorname{Re}\left(\dfrac{1}{1-z}\right) = \dfrac{1 - 2\cos\theta}{(1 - 2\cos\theta)^2 + 4\sin^2\theta}$

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{1-4\cos\theta+4\cos^2\theta+4(1-\cos^2\theta)}$$

$$= \frac{1-2\cos\theta}{1-4\cos\theta+\cancel{4\cos^2\theta}+4-\cancel{4\cos^2\theta}}$$

$$= \frac{1-2\cos\theta}{5-4\cos\theta}$$

(iii) $$\mathrm{Im}\left(\frac{1}{1-z}\right) = \frac{-2\sin\theta}{(1-2\cos\theta)^2+4\sin^2\theta}$$

$$= \frac{-2\sin\theta}{5-4\cos\theta}$$