BOARD OF STUDIES
NEW SOUTH WALES

2002
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) By using the substitution $u = \sec x$, or otherwise, find **2**

$$\int \sec^3 x \tan x \, dx \,.$$

(b) By completing the square, find $\displaystyle\int \frac{dx}{x^2 + 2x + 2}$. **2**

(c) Find $\displaystyle\int \frac{x\,dx}{(x+3)(x-1)}$. **3**

(d) By using two applications of integration by parts, evaluate **4**

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx \,.$$

(e) Use the substitution $t = \tan\dfrac{\theta}{2}$ to find **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \cos\theta} \,.$$

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 1 + 2i$ and $w = 1 + i$. Find, in the form $x + iy$,

(i) $z\overline{w}$ **1**

(ii) $\dfrac{1}{w}$. **1**

(b) On an Argand diagram, shade in the region where the inequalities **3**

$$0 \le \operatorname{Re} z \le 2 \quad \text{and} \quad |z - 1 + i| \le 2$$

both hold.

(c) It is given that $2 + i$ is a root of

$$P(z) = z^3 + rz^2 + sz + 20,$$

where r and s are real numbers.

(i) State why $2 - i$ is also a root of $P(z)$. **1**

(ii) Factorise $P(z)$ over the real numbers. **2**

(d) Prove by induction that, for all integers $n \ge 1$, **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta).$$

(e) Let $z = 2(\cos\theta + i\sin\theta)$.

(i) Find $\overline{1 - z}$. **1**

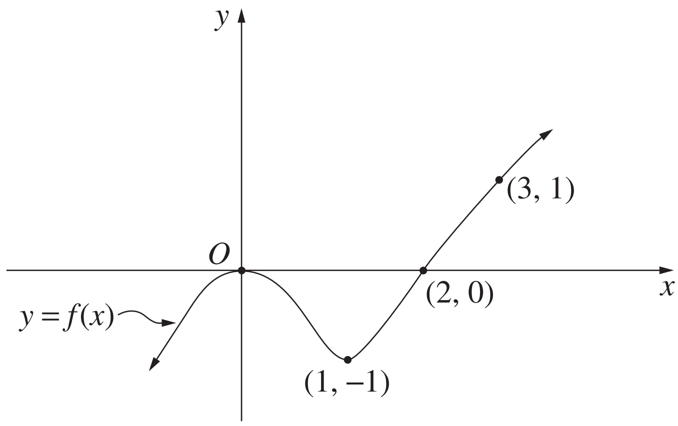
(ii) Show that the real part of $\dfrac{1}{1-z}$ is $\dfrac{1 - 2\cos\theta}{5 - 4\cos\theta}$. **2**

(iii) Express the imaginary part of $\dfrac{1}{1-z}$ in terms of θ. **1**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graph of $y = f(x)$.

Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \left| f(|x|) \right|$ **2**

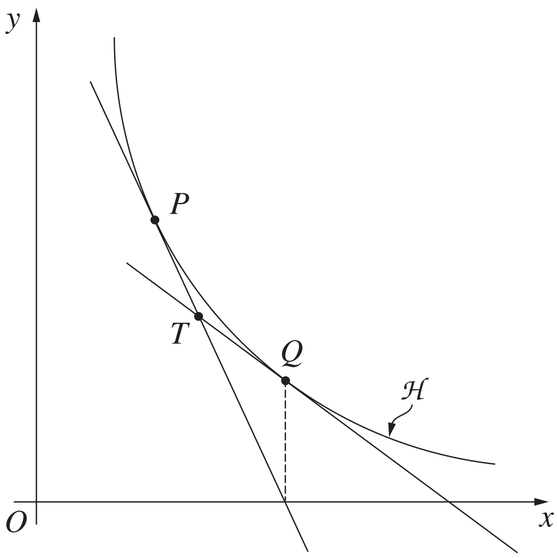
(iv) $y = \ln(f(x))$. **2**

Question 3 continues on page 5

Marks

Question 3 (continued)

(b)



The distinct points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ are on the same branch of the hyperbola $\mathcal{H}$ with equation $xy = c^2$. The tangents to $\mathcal{H}$ at P and Q meet at the point T.

(i) Show that the equation of the tangent at P is **2**

$$x + p^2 y = 2cp.$$

(ii) Show that T is the point $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$. **2**

(iii) Suppose P and Q move so that the tangent at P intersects the x axis at $(cq, 0)$. **3**

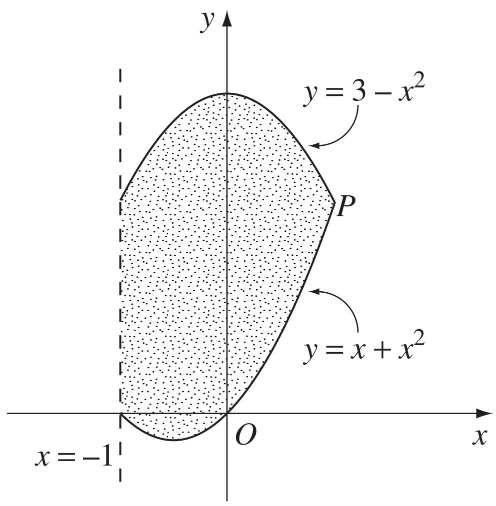
Show that the locus of T is a hyperbola, and state its eccentricity.

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a)



The shaded region bounded by $y = 3 - x^2$, $y = x + x^2$ and $x = -1$ is rotated about the line $x = -1$. The point P is the intersection of $y = 3 - x^2$ and $y = x + x^2$ in the first quadrant.

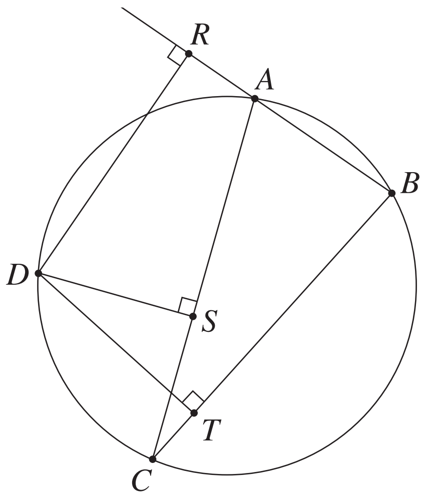
(i) Find the x coordinate of P. **1**

(ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. **3**

(iii) Evaluate the integral in part (ii). **2**

Question 4 continues on page 7

(b)



In the diagram, A, B, C and D are concyclic, and the points R, S, T are the feet of the perpendiculars from D to BA produced, AC and BC respectively.

(i) Show that $\angle DSR = \angle DAR$. **2**

(ii) Show that $\angle DST = \pi - \angle DCT$. **2**

(iii) Deduce that the points R, S and T are collinear. **2**

(c) From a pack of nine cards numbered $1, 2, 3, \ldots, 9$, three cards are drawn at random and laid on a table from left to right.

(i) What is the probability that the number formed exceeds 400? **1**

(ii) What is the probability that the digits are drawn in descending order? **2**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $4x^3 - 27x + k = 0$ has a double root. Find the possible values of k. **2**

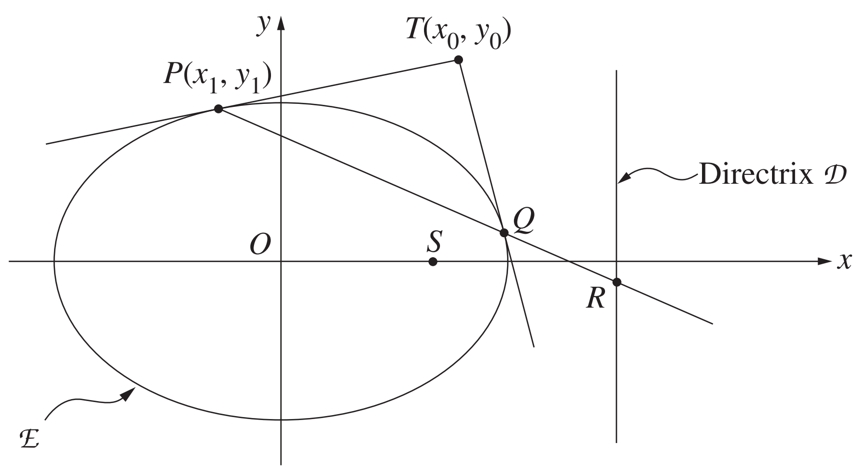
(b) Let α, β, and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha - 1$, $\beta - 1$, and $\gamma - 1$. **2**

(ii) Find a polynomial equation with integer coefficients whose roots are α^2, β^2, and γ^2. **2**

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

(c)



The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and focus S and directrix $\mathcal{D}$ as shown in the diagram. The point $T(x_0, y_0)$ lies outside the ellipse and is not on the x axis. The chord of contact PQ from T intersects $\mathcal{D}$ at R, as shown in the diagram.

(i) Show that the equation of the tangent to the ellipse at the point $P(x_1, y_1)$ is **2**

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

(ii) Show that the equation of the chord of contact from T is **2**

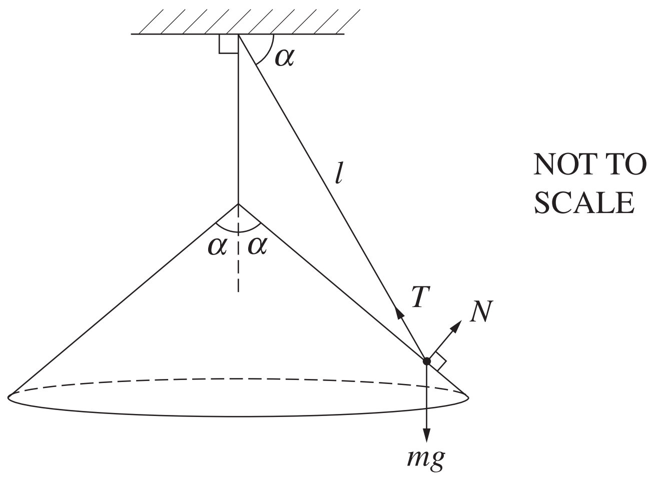
$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

(iii) Show that TS is perpendicular to SR. **3**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m is suspended by a string of length l from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity ω. The angle of the cone at its vertex is 2α, where $\alpha > \frac{\pi}{4}$, and the string makes an angle of α with the horizontal as shown in the diagram. The forces acting on the particle are the tension in the string T, the normal reaction to the cone N and the gravitational force mg.



(i) Show, with the aid of a diagram, that the vertical component of N is $N \sin \alpha$. **1**

(ii) Show that $T + N = \dfrac{mg}{\sin \alpha}$, and find an expression for $T - N$ in terms of m, l and ω. **3**

(iii) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone. Find an expression for this value of ω in terms of α, l and g. **2**

Question 6 continues on page 10

Marks

Question 6 (continued)

(b) For $n = 0, 1, 2, \ldots$ let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta .$$

(i) Show that $I_1 = \frac{1}{2} \ln 2$. **1**

(ii) Show that, for $n \geq 2$, **3**

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

(iii) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that **3**

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}.$$

(iv) By using the recurrence relation of part (ii), find I_5 and deduce that **2**

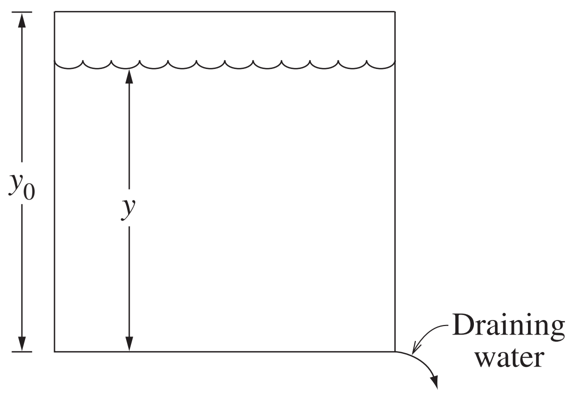
$$\frac{2}{3} < \ln 2 < \frac{3}{4} .$$

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram represents a vertical cylindrical water cooler of constant cross-sectional area A. Water drains through a hole at the bottom of the cooler. From physical principles, it is known that the volume V of water decreases at a rate given by

$$\frac{dV}{dt} = -k\sqrt{y} \,,$$

where k is a positive constant and y is the depth of water.

Initially the cooler is full and it takes T seconds to drain. Thus $y = y_0$ when $t = 0$, and $y = 0$ when $t = T$.

(i) Show that $\frac{dy}{dt} = -\frac{k}{A}\sqrt{y}$. **1**

(ii) By considering the equation for $\frac{dt}{dy}$, or otherwise, show that **4**

$$y = y_0\left(1 - \frac{t}{T}\right)^2 \quad \text{for } 0 \le t \le T.$$

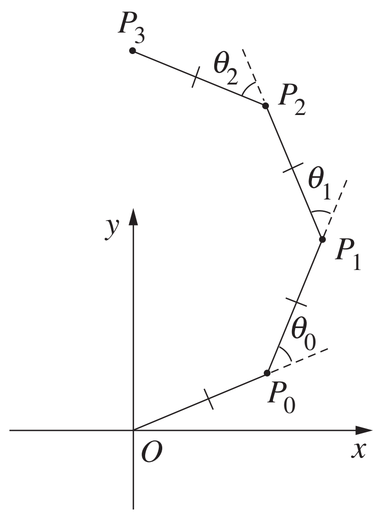
(iii) Suppose it takes 10 seconds for half the water to drain out. How long does it take to empty the full cooler? **2**

Question 7 continues on page 12

(b) Suppose $0 < \alpha,\ \beta < \frac{\pi}{2}$ and define complex numbers z_n by

$$z_n = \cos(\alpha + n\beta) + i\ \sin(\alpha + n\beta)$$

for $n = 0, 1, 2, 3, 4$. The points P_0, P_1, P_2 and P_3 are the points in the Argand diagram that correspond to the complex numbers z_0, $z_0 + z_1$, $z_0 + z_1 + z_2$ and $z_0 + z_1 + z_2 + z_3$ respectively. The angles θ_0, θ_1 and θ_2 are the external angles at P_0, P_1 and P_2 as shown in the diagram below.



(i) Using vector addition, explain why **2**

$$\theta_0 = \theta_1 = \theta_2 = \beta.$$

(ii) Show that $\angle P_0OP_1 = \angle P_0P_2P_1$, and explain why $OP_0P_1P_2$ is a cyclic quadrilateral. **2**

(iii) Show that $P_0P_1P_2P_3$ is a cyclic quadrilateral, and explain why the points O, P_0, P_1, P_2 and P_3 are concyclic. **2**

(iv) Suppose that $z_0 + z_1 + z_2 + z_3 + z_4 = 0$. Show that **2**

$$\beta = \frac{2\pi}{5}.$$

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Let m be a positive integer.

(i) By using De Moivre's theorem, show that **2**

$$\sin(2m+1)\theta = \binom{2m+1}{1}\cos^{2m}\theta\sin\theta - \binom{2m+1}{3}\cos^{2m-2}\theta\sin^3\theta + \ldots + (-1)^m\sin^{2m+1}\theta.$$

(ii) Deduce that the polynomial **3**

$$p(x) \equiv \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \ldots + (-1)^m$$

has m distinct roots

$$\alpha_k = \cot^2\left(\frac{k\pi}{2m+1}\right) \quad \text{where } k = 1, 2, \ldots, m.$$

(iii) Prove that **2**

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \ldots + \cot^2\left(\frac{m\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

(iv) You are given that $\cot\theta < \frac{1}{\theta}$ for $0 < \theta < \frac{\pi}{2}$. **2**

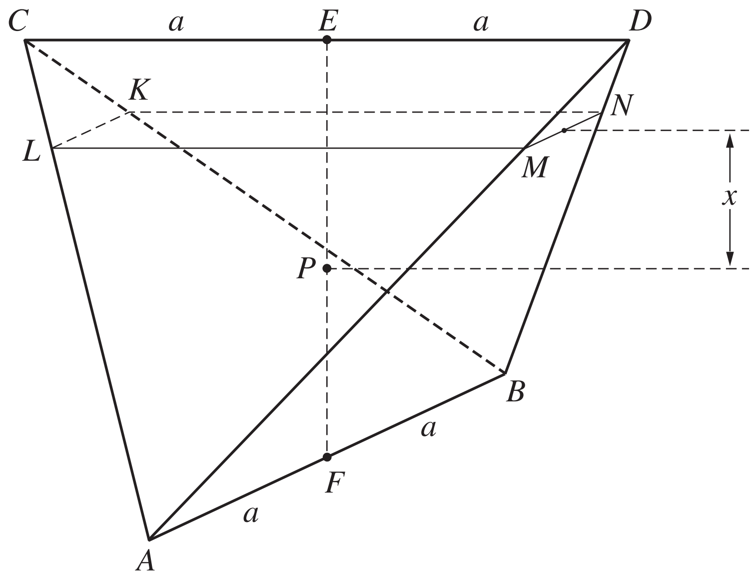
Deduce that

$$\frac{\pi^2}{6} < \left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{m^2}\right)\frac{(2m+1)^2}{2m(2m-1)}.$$

Question 8 continues on page 14

Question 8 (continued)

(b)



In the diagram, AB and CD are line segments of length $2a$ in horizontal planes at a distance $2a$ apart. The midpoint E of CD is vertically above the midpoint F of AB, and AB lies in the South–North direction, while CD lies in the West–East direction.

The rectangle $KLMN$ is the horizontal cross-section of the tetrahedron $ABCD$ at distance x from the midpoint P of EF (so $PE = PF = a$).

(i) By considering the triangle ABE, deduce that $KL = a - x$, and find the area of the rectangle $KLMN$. **4**

(ii) Find the volume of the tetrahedron $ABCD$. **2**

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$